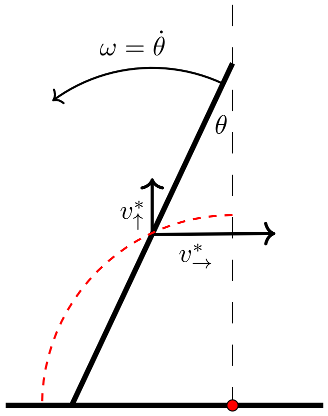
T1: Jumper (10 pts)

ENERGY APPROACH. The motion of the jumper has two phases: a push-off phase and a free vertical flight phase after the jumper loses contact with the ground. During the push-off phase, the mechanical energy of the jumper is conserved, as the forces from the ground do no work on the jumper. Let us first analyse the push-off phase.

The mechanical energy of the jumper consists of three parts: the elastic energy of the spring, $W_{el} = k(2\theta)^2/2 = 2k\theta^2$, gravitational potential energy of the jumper (relative to the ground), $W_p = 2mgy^*$ where y^* is the vertical position of the jumper's center of mass with respect to the ground; and the kinetic energy of the jumper, W_k.

The motion of the jumper consists of the motion of its center of mass and the rotation of the rods about the hinge. During the push-off phase, these motions are connected.

As the jumper is symmetric about the vertical plane through the hinge, we shall describe what happens during the push-off phase with one rod. The position of its center of mass is $(x^*, y^*) = (-\sin\theta, \cos\theta)\ell/2$.



Kinetic energy calculation 1. At the hinge, the end of the rod moves vertically with speed $2v^*_\uparrow = 2\dot y$, while the other end moves horizontally with speed $2v^*_\rightarrow = 2\dot x$. Alternatively, the rod's center of mass, located at the midpoint of the rod, moves vertically with speed $v^*_\uparrow$ and horizontally with speed $v^*_\rightarrow$, while the rod rotates about it with angular velocity $\omega = \dot\theta$. The kinetic energy of one rod is

$$W_{k,1} = \frac{1}{2}m(v^{*2}_\uparrow + v^{*2}_\rightarrow) + \frac{1}{2}J^*\omega^2$$

where $J^* = m\ell^2/12$ is the moment of inertia of the rod about the axis through its center of mass. Using $v^*_\uparrow = \dot y^* = \ell\,\sin(\theta)\cdot\dot\theta/2$ and $v^*_\rightarrow = \dot x^* = \ell\,\cos(\theta)\cdot\dot\theta/2$ kinetic energy of one rod becomes $W_{k,1} = m\ell^2\dot\theta^2/6$ and the kinetic energy of the jumper is

$$W_k = m\ell^2\dot\theta^2/3.$$

Kinetic energy calculation 2. The motion of one rod can also be described as two simultaneous rotations: the rod's center of mass moves along the circular trajectory (dashed red line in figure) of radius $l/2$ with the center at the red point with instantaneous speed $v^* = \ell\dot\theta/2$ and the rod also rotates about its center of mass with angular velocity $\omega = \dot\theta$. Instantaneous kinetic energy of one rod in these combined motions is

$$W_{k,1} = \frac{1}{2}m(\ell\dot\theta/2)^2 + \frac{1}{2}\cdot\frac{1}{12}m\ell^2\dot\theta^2 = m\ell^2\dot\theta^2/6$$

and for the two rods we get the same result for the kinetic energy of the jumper, $W_k = m\ell^2\dot\theta^2/3$. (To see, that instantaneous angular velocities of both rotations is $\dot\theta$, check how much the rod is rotated about its center of mass and what is the angle of center of mass rotation about the red point while θ changes from $\pi/2$ to 0.)

Energy conservation. With the assumption given in the problem that $k \gg mg\ell$ we may neglect the change in the gravitational potential energy of the jumper during the push-off phase: during the push-off, all the released elastic energy of the spring is transformed into kinetic energy. At any moment during the push-off phase, θ and $\dot\theta$ are related by the equation

$$2k\left(\pi^2/4 - \theta^2\right) = m\ell^2\dot\theta^2/3$$

or

$$\dot\theta^2 = \frac{6k}{m\ell^2}(\pi^2/4 - \theta^2). \tag{1}$$

During the push-off, the vertical component of the jumper's center of mass, $v^*_\uparrow$, increases until the push-off phase ends at a certain θ_0, when the jumper loses contact with the ground. (Again, we can neglect the contribution of the gravitational force to the resultant force acting on the jumper during the push-off, as it is negligible compared to the normal force from the ground.) At θ_0, both $v^*_\uparrow$ and $v^{*2}_\uparrow = \ell^2\sin^2(\theta)\cdot\dot\theta^2/4$ are maximal, so

$$\left.\frac{d(\dot\theta^2\sin^2\theta)}{d\theta}\right|_{\theta_0} = 0$$

from which we obtain a transcendental equation:

$$\theta_0\cdot\tan(\theta_0) = \pi^2/4 - \theta_0^2.$$

The solution to this equation can be found using various iterative methods, described in the next section in the context of the dynamical approach:

$$\theta_0 = 0.987 = 56.6°.$$

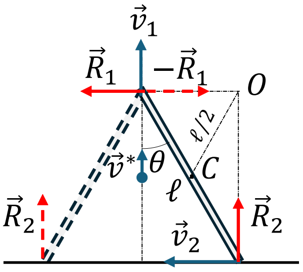
At the moment when the push-off phase ends, the vertical speed of the jumper's center of mass is $v^*_\uparrow = \frac{1}{2}\ell\sin(\theta_0)\dot\theta\big|_{\theta_0}$. During the upward flight phase, only the portion of kinetic energy corresponding to the upward translational motion of the jumper is converted into potential energy (which can no longer be neglected, since the gravitational force is the only force acting on the jumper during its flight phase). When the push-off phase ends, the translational part of the jumper's kinetic energy is

$$W_{k,t} = \frac{1}{2}2mv_{\uparrow}^{*2}\big|_{\theta_0} = \frac{3}{2}k\sin^2(\theta_0)\left(\pi^2/4-\theta_0^2\right)$$

and at the maximum height of the jump h, it is completely transformed into the jumper's potential energy $W_{p,h} = 2mgh$, so the height of the jump is

$$h = \frac{3k}{4mg}\sin^2(\theta_0)\left(\pi^2/4-\theta_0^2\right) = 1.040\cdot\frac{3k}{4mg} = 0.780\cdot\frac{k}{mg}$$

DYNAMICS APPROACH. Like in the energy approach, we consider the motion of only one of the rods. Direction of the velocity $\vec{v}_1$ of the upper end and $\vec{v}_2$ of the lower end of the rod are shown in the figure. It is evident that the instantaneous motion of any point of the rod could be represented as a pure rotation with angular velocity $\dot{\theta}$ about a horizontal axis passing through point O situated at a distance $\ell/2$ from the center C of the rod. The equation of rotational motion about O reads:



$$I_O\ddot{\theta} = \tau_O,$$

where τ_O is the net total torque of all forces applied to the rod, and

$$I_0 = m\ell^2/12 + m(\ell/2)^2 = m\ell^2/3$$

is the moment of inertia of the rod about O. From the symmetry of the system it follows that the reaction force at the hinge $\vec{R}_1$ on the right rod and its counteraction $-\vec{R}_1$ on the other rod are horizontal at any moment; therefore, $\vec{R}_1$ has no torque about O. The reaction $\vec{R}_2$ from the floor acts vertically toward O and has no torque, as well. Therefore, the only nonzero torque on the rod is due to the torsional spring $\tau_O = -2k\theta$ where the minus sign accounts for the restoring character of the torque. The torque, due to gravity could be neglected, as far as $k \gg mgl$ Finally, we get the equation of motion:

$$\ddot{\theta} = -\frac{6k}{m\ell^2}\theta,$$

which is the equation of a simple harmonic motion with angular frequency $\omega = 1/\ell\sqrt{6k/m}$ and initial conditions $\theta_0 = \pi/2$ and $\dot{\theta}_0 = 0$. The time dependence of the angle is, therefore:

$$\theta(t) = \frac{\pi}{2}\cos\omega t.$$

The velocity of the upper end of the rod is $v_1 = \ell\sin(\theta)\dot{\theta}$ and that of the center of mass of the system:

$$v^* = v_1/2 = \ell\sin(\theta)\dot{\theta}/2.$$

The equation of motion of the center of mass is $2m\dot{v}^* = 2R_2$ since we neglect the gravity. Therefore, at the moment of detachment from the floor, when $R_2 = 0$:

$$\dot{v}^* = \ell\left(\cos(\theta)\dot{\theta}^2 + \sin(\theta)\ddot{\theta}\right)/2 = 0. \qquad (2)$$

For the harmonic motion two identities hold at any moment: $\ddot{\theta} = -\omega^2\theta$ and $\dot{\theta}^2/\omega^2 + \theta^2 = \theta_0^2 = \pi^2/4$. Finally, the condition for detachment takes the form:

$$\mathrm{f}(\theta) = (\pi^2/4-\theta^2)\cos\theta - \theta\sin\theta = 0$$

which is equivalent to the equation $\theta\tan\theta = \pi^2/4-\theta^2$ obtained energetically. One of the simplest numerical approaches to that equation is the method of bisection. First, an interval $[\theta_L;\theta_R]$ has to be found where $\mathrm{f}(\theta)$ changes its sign. It is easy to establish that $\mathrm{f}(0) > 0$ and $\mathrm{f}(\pi/2) < 0$, i.e. $\theta_L = 0$ and $\theta_R = \pi/2$. Then, the interval is divided in two halves by $\bar{\theta} = (\theta_L+\theta_R)/2$. If $\mathrm{f}(\bar{\theta}) > 0$, a new $\theta_L \leftarrow \bar{\theta}$ is taken, else, $\theta_R \leftarrow \bar{\theta}$ is set. The procedure is iterated until the length of the $[\theta_L;\theta_R]$ interval becomes smaller than the acceptable uncertainty, and the final approximation is taken as the last updated $\bar{\theta}$. The procedure is ilustrated in the table below:

θ_L	$\mathrm{f}(\theta_L)$	θ_R	$\mathrm{f}(\theta_R)$	$\bar{\theta}$	$\mathrm{f}(\bar{\theta})$
0.000	2.4674	1.571	-1.5708	0.785	0.7532
0.785	0.7532	1.571	-1.5708	1.178	-0.6753
0.785	0.7532	1.178	-0.6753	0.982	0.0190
0.982	0.0190	1.178	-0.6753	1.080	-0.3390
0.982	0.0190	1.080	-0.3390	1.031	-0.1620
0.982	0.0190	1.031	-0.1620	1.006	-0.0719
0.982	0.0190	1.006	-0.0719	0.994	-0.0265
0.982	0.0190	0.994	-0.0265	0.988	-0.0037
0.982	0.0190	0.988	-0.0037	0.985	0.0076
0.985	0.0076	0.988	-0.0037	0.986	0.0019
0.986	0.0019	0.988	-0.0037	0.987	-0.0009
0.986	0.0019	0.987	-0.0009	0.987	0.0005
0.987	0.0005	0.987	-0.0009	0.987	-0.0002
0.987	0.0005	0.987	-0.0002	0.987	0.0002

The iterations obviously converge toward $\theta \approx 0.987$. Thus, at the moment of detachment, we have $\sin\theta = 0.8344$, $\dot{\theta} = \omega\sqrt{\pi^2/4-\theta^2} = 1.2220\,\omega$, so we obtain

$v^* = 1.2487\sqrt{k/m}$. Subsequently, only the kinetic energy associated with the center of mass motion is transformed into gravitational potential energy: $2\times mv^{*2}/2 = 2\times mgh$, i.e. the center of mass moves like a point-mass object projected vertically with velocity v^*. Therefore, the height of the jump is:

$$h = \frac{v^{*2}}{2g} = 0.780\,\frac{k}{mg}.$$

We have neglected the elevation $(\ell/2)\cos\theta \approx 0.276\ell$ of the center of mass at the moment of detachment, which is much smaller than h.

A functional toy, similar to the jumper, can be seen here:

T1: Marking scheme

The solution could be structured into three tasks, as specified below.

Task 1. Setting the equations for the pushup phase

	Task 1 – energy approach	Pts.
A	writes elastic energy $2k\theta^2$	0.5
B	states there are 2 phases of jump, push-off and vertical throw	0.5
C	states there is no external work during push-off phase	0.4
D	states mechanical energy during push-off is conserved	0.4
E	neglects a change of gravitational W_p during push-off	0.4
F	considers translational/rotational motion of the rods separately	0.5
G	writes W_k for the jumper correctly	1.0
H	writes equation relating θ with $\dot{\theta}$ (1) or v^* (2)	1.5
I	states that the moment of loosing contact with the ground is the moment of largest vertical speed of jumper	0.8
J	derives a trascendental equation as the condition for the largest vertical speed with derivation	1.0
Partial sum T1 task 1 - energy approach		**7.0**

	Task 1 – dynamics approach	Pts.
A	Draws the direction of velocities of the two rod ends and identifies the instantaneous center of rotation O	0.9
B	Calculates the moment of inertia $I_O = 1/3m\ell^2$ by means of Steiner's theorem	0.8
C	Comes to a conclusion that the torque of the reaction forces about O is zero	0.4
D	Neglects the torque of the force of gravity	0.2
E	Writes down the equation of rotational motion about O by accounting for the torque of the torsional spring	0.5
F	States that the equation describes a SHM and identifies its angular or linear frequency, or its period	0.5
G	Writes explicitly the time dependence $\theta(t)$ satisfying the initial conditions	1.5
H	Derives expression for v^*, either imlicitly in terms of θ and $\dot{\theta}$ or explicitly as a function of t	0.5
I	Rationalizes by means of Newton's second law that the C.M. acceleration $\dot{v}^* = 0$ in the moment of detachment	0.5
J	Derives trancedential equation for θ in the moment of detachment	1.2
Partial sum T1 task 1 – dynamics approach		**7.0**

Task 2. Solving the equation

	Problem T1 – numerical solution	Pts.
A	States or illustrates the chosen method of iterative solution: bisection, Newton, etc.	0.3
B	Indicates the starting approximation (or interval)	0.2
C	Documents sufficient number of iterations to prove the convergence of the chosen method	0.2
D	Obtains $\theta = 0.987$ to a precision of three significant digits.	0.3
Partial Sum T1 task 2 – numerical solution		**1.0**

If the student comes to the correct solution for θ in a non-systematic way (say, trial-and-error), or does not document the convergence of the chosen numerical method, no points are assigned to C and D. Also, no points are assigned to D if the final result is represented with only one or two significant digits.

Task 3. Investigating the free vertical flight

	Problem T1 - vertical flight	**Pts.**
A	Obtains the expression $v^* = 1.2487\sqrt{k/m}$ for the C.M. velocity at the moment of detachment	0.5
B	States or illustrates that the center of mass moves like a points mass of initial velocity v^*	0.2
C	Writes down the conservation of energy $mv^{*2}/2 = mgh$ for the motion after detachment or the equivalet set of equations: $h = v^*t - gt^2/2$ and $0 = v^* - gt$ describing the uniformly decelerated motion of the C.M.	0.5
D	Derives the final expression $h = 0.780k/(mg)$ with at least three significant digits of the numerical prefactor. Adding the C.M. elevation $(\ell/2)\cos\theta \approx 0.276\ell$ at the moment of detachment is not necessary but, if present, is not penalized	0.8
	Partial Sum T1 task 3 - vertical flight	**2.0**

T2: Hysteresis (10 pts)

a) Solution considering remagnetization time

The current $I(t)$ in the circuit and the voltages $U_L(t)$, $U_C(t)$ across the solenoid and the capacitor are connected through equations

$$\begin{aligned} I(t) &= C\frac{\mathrm{d}U_C}{\mathrm{d}t} = -C\frac{\mathrm{d}U_L}{\mathrm{d}t} \\ U_L(t) &= NA\frac{\mathrm{d}B}{\mathrm{d}t} = -U_C(t) \end{aligned} \tag{3}$$

where $A = \pi R^2$. The magnetic field strength in the solenoid is

$$H = \frac{NI}{\ell}\,. \tag{4}$$

At the start there is a large current I_i in the circuit such that $H(I_\mathrm{i}) \gg H_0$. Initially, the current charges the capacitor, while the current $I(t)$ and the field $H(t)$ decrease. The point, representing the momentary combination of $H(t)$ and $M(t)$ moves along the top horizontal line in the hysteresis curve (to the left). The magnetization M does not change during this phase even when I crosses 0 and becomes negative. In this phase $\frac{\mathrm{d}B}{\mathrm{d}t} = \mu_0\frac{\mathrm{d}H}{\mathrm{d}t}$. As in an ideal LC-circuit, energy is conserved.

When the current reaches the critical magnitude in the opposite direction, i.e.

$$I = -I_0 = -\frac{H_0\ell}{N}\,, \tag{5}$$

the magnetization M starts to reverse. The vertical hysteresis curve at H_0 means that during the magnetization reversal (which takes some time) H stays constant and equal to H_0. Therefore, also the current stays constant and equal to $-I_0$, while M changes. For the critical current we obtain

$$-I_0 = C\frac{\mathrm{d}U_C}{\mathrm{d}t} \tag{6}$$

$$U_C(t) = -NA\mu_0\frac{\mathrm{d}M}{\mathrm{d}t} \tag{7}$$

Solving these equations with $t = 0$ corresponding to the moment I reaches $-I_0$ (when the orientation of magnetization in the core starts to change) and using $U_1 = U_C(0)$ gives

$$U_C(t) = U_1 - \frac{I_0}{C}t \tag{8}$$

$$M(t) = M_0 - \frac{1}{NA\mu_0}\left(U_1 t - \frac{I_0}{2C}t^2\right) \tag{9}$$

After the time interval Δt the core is completely remagnetized with $M(\Delta t) = -M_0$. We get

$$\Delta t = \frac{C}{I_0}\left(U_1 - \sqrt{U_1^2 - \frac{4NA\mu_0 I_0 M_0}{C}}\right) \tag{10}$$

The corresponding voltage is

$$U_2 = U_C(\Delta t) = \sqrt{U_1^2 - \frac{4NA\mu_0 I_0 M_0}{C}}. \tag{11}$$

The energy needed to change the orientation of magnetization of the core thus is

$$\frac{1}{2}C(U_1^2 - U_2^2) = 2NA\mu_0 I_0 M_0 = 2A\ell\mu_0 H_0 M_0 \tag{12}$$

After complete remagnetization the circuit behaves again like an LC-circuit, but with lower energy. From Δt on the current changes harmonically until it reaches the critical value I_0, and the story repeats. Since the energy needed for the magnetization reversal process is independent of the starting voltage (as long as it is fully completed), during one full cycle the energy loss is twice the value computed above, i.e. $\Delta E = 4A\ell\mu_0 H_0 M_0$. The maximum current I_2 after one cycle is reduced from I_i due to the lower energy with $\Delta E = \frac{1}{2}L(I_\mathrm{i}^2 - I_2^2)$, where L is the inductance $L = \mu_0 N^2 A/\ell$ of the solenoid. It follows

$$I_2 = \sqrt{I_\mathrm{i}^2 - 8\frac{A}{L}\ell\mu_0 H_0 M_0} = I_\mathrm{i}\sqrt{1 - \frac{8\ell^2 H_0 M_0}{N^2 I_\mathrm{i}^2}} \tag{13}$$

Note that $NI_\mathrm{i}/\ell = H(I_\mathrm{i})$. Since $H(I_\mathrm{i}) \gg H_0$ we can approximate the change ΔI in the maximum current to

$$\begin{aligned}\Delta I = I_2 - I_\mathrm{i} &= I_\mathrm{i}\sqrt{1 - \frac{8\ell^2 H_0 M_0}{N^2 I_\mathrm{i}^2}} - I_\mathrm{i} \\ &\approx -\frac{4\ell^2 H_0 M_0}{N^2 I_\mathrm{i}} = -4I_\mathrm{i}\frac{H_0 M_0}{H(I_\mathrm{i})^2}\end{aligned} \tag{14}$$

Solution using energy considerations

Originally the energy used to remagnetize the core is stored in the inductor, so $E_0 = \frac{1}{2}LI_\mathrm{i}^2$. After a single cycle, the energy is $E_2 = \frac{1}{2}LI_2^2$, where the same notation as in the other approach is used. The inductance of the solenoid is given by $L = \mu_0 N^2 A/\ell$. The loss of stored energy during one cycle is proportional to the area of the square in the hysteresis curve (an explanation is provided below) with $\Delta E = 4A\ell\mu_0 H_0 M_0$.

We thus get

$$I_2 = I_\mathrm{i}\sqrt{1 - \frac{8\ell^2 H_0 M_0}{N^2 I_\mathrm{i}^2}} \tag{15}$$

from which the result (14) follows as above.

To understand the relation between the energy loss during magnetization and the area of the square in the hysteresis curve, we consider the instantaneous electrical power in the solenoid and integrate it over over one cycle. Using equations (3) and (4) we have

$$\begin{aligned}P = U_L I = ANI\frac{\mathrm{d}B}{\mathrm{d}t} &= AH\ell\frac{\mathrm{d}B}{\mathrm{d}t} \\ &= \mu_0 AH\ell\left(\frac{\mathrm{d}H}{\mathrm{d}t} + \frac{\mathrm{d}M}{\mathrm{d}t}\right)\end{aligned} \tag{16}$$

Since the circuit is isolated and the capacitor uncharged at the beginning of an oscillation cycle the total electrical work over one cycle, starting at t_i and ending at t_e should equal zero. Integrating the power over the cycle gives

$$\begin{aligned}0 &= \int_{t_\mathrm{i}}^{t_\mathrm{e}} P(t)\mathrm{d}t \\ &= \mu_0 A\ell\left[\int_{t_\mathrm{i}}^{t_\mathrm{e}} H(t)\frac{\mathrm{d}H}{\mathrm{d}t}\mathrm{d}t + \int_{t_\mathrm{i}}^{t_\mathrm{e}} H(t)\frac{\mathrm{d}M}{\mathrm{d}t}\mathrm{d}t\right]\end{aligned} \tag{17}$$

The first integral gives $\frac{1}{2}H(t_\mathrm{e})^2 - \frac{1}{2}H(t_\mathrm{i})^2$. The second integral gives $4H_0M_0$, from which we get

$$\Delta E = \frac{1}{2}L\left(I(t_\mathrm{i})^2 - I(t_\mathrm{e})^2\right) = 4A\ell\mu_0 H_0 M_0 \tag{18}$$

Note: The quantity $\mathrm{d}u = H\mathrm{d}B$ is the change per unit volume of the magnetic energy of the solenoid.

b) As long as the current amplitude exceeds the critical current required to change the magnetization orientation, the circuit will continue loosing energy in the remagnetization process. After many cycles the voltage at the start of the remagnetization process is too small for the remagnetization to be fully completed. The remagnetization process during the phase of the constant current I_0 stops when the voltage (and charge in capacitor) becomes zero. In this case no more energy is stored in the capacitor that can be used for remagnetization. The core then remains partially magnetized and there is no more energy dissipation connected with the core.

From now on, the circuit behaves like an LC circuit with current amplitude of $I_0 = \frac{H_0\ell}{N}$. In the $H-M$ graph this corresponds to moving back and forth on a horizontal line within the square. Therefore, the maximum current after many oscillations is the amplitude of the current oscillations in the LC-circuit

$$I_\mathrm{max} = I_0 = H_0\frac{\ell}{N}. \tag{19}$$

c) The two distinct phases are the LC-circuit harmonic behaviour (phase A) and the remagnetization process (phase B). To find the maximum duration for the latter let us first consider the duration of the magnetization processes for the case where full magnetization occurs. The duration was computed in (10) and equals

$$\Delta t = \frac{C}{I_0}\left(U_1 - \sqrt{U_1^2 - \frac{4NA\mu_0 I_0 M_0}{C}}\right) \tag{20}$$

Since I_0 is the same and constant for each magnetization process, we have to find the minimal value of Δt over all possible U_1. We can compute a derivative of Δt with respect to U_1,

$$\frac{\mathrm{d}\Delta t}{\mathrm{d}U_1} = \frac{C}{I_0}\left(1 - \frac{U_1}{\sqrt{U_1^2 - 4NA\mu_0 I_0 M_0/C}}\right) < 0. \tag{21}$$

The derivative is negative for all values of U_1, which means that Δt is monotoneously decreasing with increasing U_1. Thus the voltage has to be as small as possible. On the other hand, when the voltage at the start of the cycle is too small, it drops to zero before the core is fully remagnetized. It follows from equation (6) that this happens after time

$$\Delta t = \frac{U_1 C}{I_0} \tag{22}$$

Hence for maximal duration Δt the voltage U_1 should be as large as possible. Together, the optimal initial voltage is the one where the voltage drops to zero precisely in the moment when the core is fully magnetized. The corresponding U_1 can be found for instance using (10) as the smallest U_1 for which a real-valued Δt exists, which is

$$U_1 = 2\sqrt{\frac{NA\mu_0 I_0 M_0}{C}} \tag{23}$$

This gives the maximum duration as

$$\begin{aligned}\Delta t_{\max} &= 2\sqrt{\frac{NA\mu_0 C M_0}{I_0}} \\ &= 2\sqrt{\frac{N^2 A\mu_0 C M_0}{\ell H_0}} = 2\sqrt{LC\frac{M_0}{H_0}}\end{aligned} \tag{24}$$

Marking scheme T2

Problem T2 a)		**Pts.**
A	Relate energy loss to capacitor voltage drop (12) *or* area under H-M-curve. Stating only $E_c = \frac{1}{2}CU^2$ gives 0.3 pts	0.4
B	Determine energy loss over full cycle from capacitor voltage drop *or* area under H-M-curve (-0.4 pts if only half cycle is considered or numerical factor is erroneous)	1.0
C	Relate current to energy $\frac{1}{2}LI^2$ stored in inductor	0.3
D	State/Derive/Use $L = \mu_0 N^2 A/\ell$ for inductance of solenoid	0.3
E	Determine change in current (14) with or without approximation	1.0
Partial sum T2 a)		**3.0**

Problem T2 b)		**Pts.**
A	State/Derive/Use $H(I) = \frac{NI}{\ell}$	0.3
B	Realize LC-circuit behaviour, when no remagnetization occurs	0.5
C	Realize that energy is lost only in remagnetization	0.5
D	Determine critical current I_0 (19)	0.5
E	Realize that current amplitude after many oscillations equals critical current	0.5
Partial sum T2 b)		**2.3**

Problem T2 c)		**Pts.**
A	Identify phase B as remagnetization process	0.3
B	State current and voltage equations (3) (0.3 pts. each, only 0.1 pts. each for $Q = C/U$ and $U = \mathrm{d}\phi/\mathrm{d}t$)	0.6
C	Realize that current stays at critical current I_0 during remagnetization	0.5
D	Determine M as function of time (9)	0.5
E	Determine time interval for remagnetization (10)	0.5
F	For complete remagnetization - Derive that duration is maximised for minimal possible voltage	0.5
G	For incomplete remagnetization - Determine time for complete discharge of capacitor	0.5
H	For incomplete remagnetization - Derive that duration is maximised for maximal possible voltage	0.3
I	Compute optimal value for voltage (23)	0.5
J	Determine maximum duration (24)	0.5
Partial sum T2 c)		**4.7**

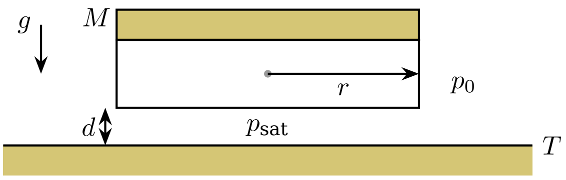
General rules for marking in T2:

- **Depending on the approach taken, points may be awarded also for subquestions not directly.** So make sure to check all aspects of the marking scheme even if students only worked e.g. on part a).
- The grain size for marking is 0.1 Pts.
- Partial marks can be awarded for most aspects.
- For each mistake in calculation (algebraic or numeric) 0.2 Pts. are deducted.
- If a mistake leads to a dimensionally incorrect expression no marks are given for the result.
- Propagating errors are not punished again unless they are dimensionally wrong or entail oversimplified/wrong physics (e.g. using an approximation in a regime it does not apply).

T3: Skating (10 pts)

Part a

Summary: The friction force on the sliding puck behaves differently depending on whether ΔT is larger or smaller than a threshold ΔT_c. When ΔT is sufficiently large (i.e. the plate is hot enough), sublimation occurs on the lower surface of the puck. The gaseous CO_2 formed by sublimation pushes the puck upwards, creating a gap between the puck and the plate. The friction in this regime is relatively low. If ΔT is too small, sublimation does not occur on the lower surface. In this regime, the puck remains in contact with the plate and the dry friction between the puck and the plate is large and independent of ΔT. Thus, ΔT_c is the temperature above which sublimation can occur and create a gap.



Calculation: Suppose ΔT is slightly larger than ΔT_c, so that sublimation occurs but the resulting gap is extremely narrow. Such a narrow layer of CO_2 gas provides negligible thermal insulation between the puck and the disc. Therefore, the temperature of the gas and the lower surface of the puck must be close to $T_s + \Delta T_c$, the temperature of the plate.

For sublimation to occur on the lower surface at this temperature, the pressure of the CO_2 gas in contact with this surface must be the saturated vapour pressure for $T = T_s + \Delta T_c$. This saturated vapour pressure, which we denote by $p_0 + \Delta p_c$, can be found from the Clausius-Clapeyron relation given in the question:

$$\frac{\mathrm{d}p_{\text{sat}}}{\mathrm{d}T} = \frac{\mu\lambda p_{\text{sat}}}{RT^2}. \tag{25}$$

On the other hand, Δp_c must also be such that the upwards force on the puck due to the excess pressure of the gas in the gap (above p_0) balances the weight of the puck. The dry ice mass $\pi r^2 h\rho \approx 0.5\,\mathrm{g}$ is much smaller than the mass of the metal disc, $M = 10\,\mathrm{g}$. Therefore, $\pi r^2 \Delta p_c \approx Mg$. This implies $\Delta p_c/p_0 \approx 3\cdot 10^{-3} \ll 1$, so (25) gives $\Delta T_c/T_s \sim (RT_s/\mu\lambda)(\Delta p_c/p_0) \ll 1$ (since $RT_s/\mu\lambda \approx 0.1$ is not large). Therefore, (25) reduces to

$$\frac{\Delta p_c}{\Delta T_c} \approx \frac{\mu\lambda p_0}{RT_s^2}, \tag{26}$$

so we find

$$\Delta T_c \approx \frac{R}{\mu\lambda}\frac{MgT_s^2}{p_0\pi r^2} \approx 40\,\mathrm{mK}. \tag{27}$$

Alternatively, if we ignore the order-unity coefficient ($1/\pi$ in this case), $\Delta T_c \approx RMgT_s^2/\mu\lambda p_0 r^2 \approx 120\,\mathrm{mK}$. Throughout the rest of this solution, we will ignore order-unity coefficients (like π) in our estimates.

Extra detail

In this problem, we must assume there is no sublimation on the upper surface of the dry ice i.e. between the dry-ice disc and the metal disc. This is realistic because the temperature gradient across the gap ($\sim \Delta T/d$) is much greater than the temperature gradient between the upper and lower surfaces of the dry-ice disc, ΔT_{ice}, that would exist if sublimation were occurring at both surfaces. As a result, the heat flux upwards through the dry ice is negligible compared to the heat flux through the gap, so insufficient thermal energy will flow to the top surface to cause another gaseous layer to form.

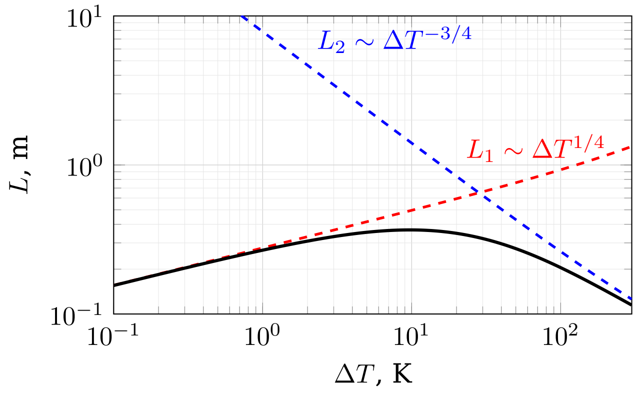
We can estimate ΔT_{ice} using the fact that, if sublimation is occurring at the upper surface, then the gas layer there must be at a pressure below the pressure in the bottom layer by $\Delta p_{\text{ice}} \approx \rho h g$, the pressure required to support the dry-ice disc. The temperature difference would then be $\Delta T_{\text{ice}} \approx T_s(RT_s/\mu\lambda)(\Delta p_{\text{ice}}/p_0)$ using (25). But this is only $\Delta T_{\text{ice}} \approx 0.002\,\mathrm{K}$.

Part b

Summary: As ΔT increases above ΔT_c, the rate of sublimation increases. More CO_2 gas is created and the gap between the puck and the plate widens. The gas needs to flow out of the gap, and calculating the resulting flow (treating the CO_2 gas as a fluid) will allow us to determine the gap width d as a function of ΔT. When the puck slides, the flow beneath it has a non-zero shear. The viscous force due to this shear is what slows the puck down. If ΔT is not too large, the viscous friction is large and the puck will come to a stop well before the dry ice has fully sublimated. We will refer to this regime as the 'low-temperature regime', $L \approx L_1(\Delta T)$.

When ΔT increases, d increases and the viscous friction decreases, which means $L_1(\Delta T)$ is an increasing function of ΔT. However, as ΔT increases, the dry ice also sublimates faster. Once ΔT is sufficiently large, the final displacement of the metal disc is determined by the distance the puck travels before the dry ice evaporates completely. When there is no dry ice remaining, the metal disc drops down onto the plate and moves no further (as we assume a relatively large dry friction between the disc and the plate). We will refer to this range of ΔT as the 'high-temperature regime', $L \approx L_2(\Delta T)$.

We will find that the distance $L(\Delta T)$ is maximised approximately at the value ΔT_m where the two regimes meet: $L_{\max} \approx L_1(\Delta T_m) = L_2(\Delta T_m)$.



Calculation: First we estimate $L_2(\Delta T)$ in the high-temperature regime. For this, we need to know the rate of dry-ice sublimation for a given ΔT. Sublimation requires energy; this energy is supplied by a flow of heat from the hotter plate to the cooler dry ice. The heat is conducted through the CO_2 gas in the gap.

The temperature of the gas in the gap will, to a good approximation, vary linearly with height between $T_s + \Delta T$ at the plate and T_s at the lower surface of the puck. In reality, the temperature at the lower surface of the puck will deviate slightly from T_s because the pressure of the CO_2 gas in the gap is slightly higher than p_0, as in part a), and varies with position, as explained after (35). However, by similar reasoning to that of part a), the size of this deviation is $\sim \Delta T_c$. We will find that $\Delta T_m \gg \Delta T_c$, so this deviation can be neglected. Then, the small width of the gap ensures that the temperature profile is similar to what we would find in the gap between two infinite planes held at different temperatures, which is a linear profile. The vertical temperature gradient in the gap is therefore $\sim \Delta T/d$, which means there is an upwards heat flux in the gap of size $\sim \kappa \Delta T/d$. Note that the fact that the temperature profile is linear is not essential here; what matters for these estimates is that the gradient is set by ΔT and the gap width. The energy flowing into the dry ice per unit time is $\sim \kappa \Delta T\,(r^2/d)$, by integrating the heat flux over the area of the lower surface. Therefore, CO_2 gas in the gap is produced by sublimation at a rate (mass per unit time) of

$$\dot{m} \sim \frac{\kappa}{\lambda} \Delta T \frac{r^2}{d}, \tag{28}$$

since sublimation at the sides of the dry-ice disc can be neglected. The time τ taken for all the mass of the dry ice to sublimate is

$$\tau \sim \frac{r^2 h \rho}{\dot{m}} \sim \frac{\rho\lambda}{\kappa} \frac{hd}{\Delta T}, \tag{29}$$

using (28). Thus, the distance that the puck slides is

$$L_2(\Delta T) \sim v \frac{\rho\lambda}{\kappa} \frac{hd}{\Delta T}. \tag{30}$$

Note that this estimate assumes the puck speed does not fall significantly below v on the sublimation timescale. This assumption is valid when $\Delta T \gg \Delta T_m$, which defines the high-temperature regime (we show this more explicitly in (42)).

Next we consider the low-temperature regime $0 < \Delta T \ll \Delta T_m$. This regime is easiest to understand if we first discuss what happens when the disc is not given any horizontal velocity.

Without any horizontal velocity, the system is symmetric under rotations about the axis of the puck (axisymmetric). The flow of the CO_2 out from underneath the puck is then radial and axisymmetric. If the dry ice sublimates slowly ($\dot{m}/\rho r^2 h \ll u/r$), the flow will be close to a steady flow at all times (the process is quasi-stationary). It will be useful to estimate the typical speed u of the flow. At steady state, all the CO_2 gas sourced beneath the puck must flow out at the rate that it is created. The rate at which mass flows out through the outer radial edge of the gap (which is a cylindrical surface of surface area $\sim rd$) is $\sim \rho_{CO_2} urd$, where ρ_{CO_2} is the mass density of the CO_2 gas. This must equal $\dot{m}$, so $u \sim \dot{m}/\rho_{CO_2} rd \sim (\kappa/\rho_{CO_2}\lambda)(r/d^2)\,\Delta T$ using (28). The density is given by the ideal-gas law $\rho_{CO_2} \sim \mu p_0/RT_s$, which means

$$u \sim \frac{\kappa}{\mu\lambda} \frac{RT_s}{p_0} \frac{r}{d^2} \Delta T. \tag{31}$$

We took the temperature to be T_s in the ideal-gas law because the temperature of the gas and puck will be close to T_s when $\Delta T_m \ll T_s$. This also means we can approximate the density of the gas in the gap as a constant (in reality, the linear temperature profile in the gap will cause some variation of density with height, but this is negligible when $\Delta T_m \ll T_s$). For estimate (31) to be useful, we need to find another relation involving u. Such a relation will be provided by the balance between pressure and viscosity in the gas.

Since the gap width is small, $d \ll r$, the Reynolds number of the flow is very low and the flow is similar to the Poiseuille flow of a viscous fluid between two infinite planes. The very low Reynolds number means the injection of CO_2 gas due to sublimation at the dry-ice surface will have a small effect on the velocity profile, provided the gas is not injected with an unrealistically large normal velocity. Then, the characteristic size of the flow is

$$u \sim \frac{\Delta p}{\eta} \frac{d^2}{r}, \tag{32}$$

as in a Poiseuille flow between two infinite planes. This can also be obtained from dimensional analysis, or via the following more advanced derivation.

Treating the shear viscosity η as a constant (in reality, the viscosity of a gas depends on temperature, but the temperature variation is small, $\Delta T_m \ll T_s$), the radial flow speed $u(\varrho, z)$ satisfies

$$\frac{\mathrm{d}p}{\mathrm{d}\varrho} \approx \eta \frac{\partial^2 u}{\partial z^2} \tag{33}$$

in cylindrical coordinates (ϱ, φ, z) (this is known as the 'lubrication', or 'thin film', approximation). Equation (33) says that a steady flow results from a balance between radial pressure gradients and viscous forces in the fluid. We have used the fact that the pressure $p(\varrho)$ is approximately independent of z, as it would be in a Poiseuille flow between two infinite planes. The solution to (33) is

$$u(\varrho, z) = -\frac{1}{2\eta}\frac{\mathrm{d}p}{\mathrm{d}\varrho} z(d-z), \tag{34}$$

where we use no-slip boundary conditions at the surfaces at $z = 0$ or $z = d$. Note that, as required by axisymmetry, $u(\varrho, z)$ vanishes at $\varrho = 0$, where p has a local maximum so $\mathrm{d}p/\mathrm{d}\varrho = 0$.

For the estimates in this problem, equation (33) and its solution are not necessary. All that is required is the fact that the characteristic size of the flow satisfies (32).

Extra detail

In order for the gas in the gap to be accurately modelled by fluid equations like (33), the gap cannot be too small. The first reason is that molecules of the gas must experience many collisions with each other in the time it would take for them to traverse the gap. Molecular collisions are the microscopic cause of viscosity and the reason heat is transported by conduction (instead of being transported by hot molecules travelling, uninterrupted, from the hotter plate to the puck). The mean free path is $\lambda_{\text{mfp}} \sim 1/n_{\text{CO}_2} r^2_{\text{CO}_2}$, where n_{CO_2} is the CO_2 number density and r_{CO_2} is an approximate radius of a CO_2 molecule. The gap must be large enough that $d \gg \lambda_{\text{mfp}}$. Second, the gap must be large enough that the surfaces can be treated as smooth rather than rough: d must be larger than the characteristic height of any 'bumps' in the surfaces. We are instructed to assume this in the problem.

How large is Δp? As in part a, the excess pressure must support the weight of the puck, so

$$\Delta p \sim \frac{Mg}{r^2}. \tag{35}$$

Note that, in part a, Δp was uniform over the lower surface of the puck. In this part, $\Delta p(\varrho)$ decreases with ϱ, because the radial pressure gradient must drive the outflow of CO_2 gas. As a result, the pressure at the dry-ice surface is larger at smaller ϱ, and (according to Clausius-Clapeyron) the temperature $T_s + \Delta T(\varrho)$ at the dry-ice surface is also larger at smaller ϱ.

Combining (32) and (35), we find

$$u \sim \frac{Mg}{\eta}\frac{d^2}{r^3}. \tag{36}$$

Then, substituting (31) into (36) gives

$$d \sim \left(\frac{\eta\kappa}{\mu\lambda}\frac{RT_s}{p_0}\frac{\Delta T}{Mg}\right)^{\frac{1}{4}} r. \tag{37}$$

This is the relation that determines the gap width as a function of ΔT.

Let us now return to the situation where the puck is given an initial horizontal speed v. The flow velocity of the gas beneath the puck must change, because the radial flow (34) no longer satisfies the no-slip boundary condition at $z = d$. Fortunately, we can correct this by simply adding the linear shear flow $u_2(z) = vz/d$ (in the direction of v) to the axisymmetric flow discussed above (i.e. to the flow described by (34)). The resulting combination satisfies the no-slip boundary conditions at each surface. This simple superposition is possible because fluid flow in a low-Reynolds number regime is a linear phenomenon (the governing differential equation (33) is linear).

The new flow exerts a viscous force on the lower surface of the puck, of size $\sim \eta v r^2/d$. Therefore, the equation of motion for the puck takes the form

$$M\frac{\mathrm{d}v}{\mathrm{d}t} \approx -\eta\frac{r^2}{d}\frac{\mathrm{d}x}{\mathrm{d}t}. \tag{38}$$

Integrating over time, we find

$$L_1(\Delta T) \sim \frac{Mv}{\eta}\frac{d}{r^2}. \tag{39}$$

Note that $L_2(\Delta T)$ in the high-temperature regime increases as the temperature decreases, while $L_1(\Delta T)$ in the low-temperature regime increases as the temperature increases. The temperature at which $L(\Delta T)$ is maximised is therefore the temperature $\Delta T \approx \Delta T_m$ at which the two regimes meet, so that neither (30) nor (39) is valid. Equating (30) and (39), we find

$$\Delta T_m \sim \frac{\eta\rho\lambda}{\kappa}\frac{hr^2}{M} \sim 9\,\mathrm{K}. \tag{40}$$

Interestingly, ΔT_m is independent of d. Therefore, result (40) could be derived without solving for the radial flow and obtaining (37). To calculate $L_{\max}$, we can substitute (40) into either (30) or (39); we constructed ΔT_m so that the result would be the same either way. Using, in addition, equation (37) for d, we find

$$L_{\max} \sim v\left(\frac{\rho}{\mu\eta^2}\frac{RT_s}{p_0}\frac{M^2}{g}\frac{h}{r^2}\right)^{1/4} \sim 1.7\,\mathrm{m}. \tag{41}$$

One could equally use an 'average' temperature such as $T_s + \Delta T_m/2$ to estimate ρ_{CO_2}, which would give a slightly higher value of $L_{\max}$.

Alternatively, instead of using $L_1(\Delta T_m) = L_2(\Delta T_m)$, we can say that around the maximum $\Delta T = \Delta T_{\max}$, the puck still slows according to $\mathrm{d}L/\mathrm{d}t = v\exp[-(\eta r^2/Md)t]$ due to viscous friction, but it must stop moving when the dry ice has fully sublimated. This occurs at time $\tau(\Delta T) = \rho\lambda h d/\kappa\Delta T$, according to (29). Thus

$$\begin{aligned} L(\Delta T) &= v\int_0^{\tau(\Delta T)} \exp\left(-\frac{\eta r^2}{Md}t\right)\mathrm{d}t \\ &= \frac{vMd}{\eta r^2}\left[1-\exp\left(-\frac{\eta r^2\rho\lambda h}{M\kappa\Delta T}\right)\right] \\ &= L_1(\Delta T)\left(1-e^{-\Delta T_m/\Delta T}\right). \end{aligned} \tag{42}$$

Since $L_1(\Delta T) \sim \Delta T^{1/4}$, there is no elementary analytical formula for the maximum value of $L(\Delta T)$. But we can still estimate that the maximum happens around $\Delta T \approx \Delta T_m$ and $L_{\max} \approx L_1(\Delta T_m)(1-e^{-1}) \approx L_1(\Delta T_m)$, which is consistent with the previous estimate.

General rules for marking T3:

- Marks for one part of the problem should be awarded even if it is contained in a student's working for another part. For example, make sure to check whether students can obtain marks for part b) even if they only worked on part a) (the only exception is item T3 b) L).
- Similarly, students do not need to obtain points in the same order as listed in the mark scheme or solution. They can, for example, calculate ΔT_m before they calculate d, or the other way around.
- Estimates do not require correct order-unity coefficients (such as π or 2); what is required is the correct scaling with physical quantities. When numbers are computed, the mark scheme provides the acceptable range in the format $X_{\min}\ldots X_{\max}$.
- If students obtain an estimate in the mark scheme in a rearranged form (e.g. they write down ΔT in terms of d instead of d in terms of ΔT), they obtain full credit.
- For part a) item F and part b) item Q, students who obtain a correct formula but make a mistake when substituting numbers into it receive 0.4 points.

Marking scheme T3

Problem T3 a) 2pts		**Pts:**
A	State/use that L is negligible and independent of ΔT when the puck and plate are in direct physical contact	0.2
B	Use $T_s + \Delta T_c$ for the puck temperature	0.3
C	Realise that Clausius-Clapeyron must be satisfied at the lower surface (argument about extremely small gap does not need to be explicit)	0.3
D	Use Clausius-Clapeyron to connect Δp_c and ΔT_c (or in integrated form)	0.2
E	Estimate the excess pressure in the gas using $\pi r^2\Delta p_c \approx Mg$	0.3
F	Numerical value $\Delta T_c = 3\,\mathrm{mK}\ldots 300\,\mathrm{mK}$	0.7
Problem T3 b) 8pts		**Pts:**
High temperature regime, 2pts		
A	State/use that $L(\Delta T)$ can be limited by complete sublimation of the dry ice	0.5
B	Temperature gradient in gap is $\sim \Delta T/d$, or heat flux in gap is $\sim \kappa\Delta T/d$	0.3
C	Obtain $\dot{m} \sim (\kappa/\lambda)\,\Delta T\,(r^2/d)$	0.4
D	Obtain $L(\Delta T) \sim v(\rho\lambda/\kappa)(hd/\Delta T)$	0.7
E	State/use that the temperature on the lower puck surface is $\sim T_s + \Delta T_c$	0.1
Low temperature regime, 2pts		
F	State/use that $L(\Delta T)$ can be limited by friction from viscosity	0.5
G	Estimate $\dot{m} \sim \rho_{\mathrm{CO}_2} urd$	0.3
H	Estimate $\rho_{\mathrm{CO}_2} \sim \mu p_0/RT_s$ (or any reasonable pressure/temperature on RHS)	0.2
I	State/use that friction force is $F_f \sim \eta v r^2/d$	0.3
J	Obtain $L(\Delta T) \sim (Mv/\eta)(d/r^2)$.	0.7
Intersection of regimes, 4pts		
K	Estimate $u \sim (\Delta p/\eta)(d^2/r)$	0.7
L	State/use $\Delta p \sim Mg/r^2$ still holds in part b)	0.3
M	State/use no-slip boundary conditions apply, or state parabolic solution for $u(r,z)$	0.3
N	Obtain $d \sim [(\eta\kappa/\mu\lambda)(RT_s/p_0)(\Delta T/Mg)]^{1/4}\,r$	0.7
O	Idea to estimate $L_{\max}$ from the point where the regimes meet or overlap **or** integrated $L \sim L_1(1-e^{\Delta T_m/\Delta T})$ with both effects taken into account	0.6
P	Obtain $\Delta T_m \sim (\eta\rho\lambda/\kappa)(hr^2/M)$ or a numerical value for $\Delta T_m = 1\,\mathrm{K}\ldots 100\,\mathrm{K}$	0.7
Q	Numerical value for $L_{\max} = 0.1\,\mathrm{m}\ldots 10\,\mathrm{m}$	0.7