Solutions

General grading instructions:

- Common small penalties
  - Missing units in measurements, only penalized once per subtask. **-0.1 pts**
  - Solution in different units than asked: **-0.1 pts**
  - Minor arithmetic error: **-0.1 pts**
  - When graphically evaluating data - using less than half the area on graph paper for plots: **-0.1 pts**
- When grading numerical results, we do not award points for correct values achieved by guessing (e.g. just stating $n = 1.35$), without a effective measurement method presented and raw data recorded.
- When determining the parameters from a linear fit, both a graphical solution as well as using a linear regression on a permitted calculator can be awarded full points for data evaluation. However in the latter case, it is necessary to state that a linear regression on a calculator was used and all raw data used must be presented in a table on paper.
- For all items in a grading scheme, we only award either the full points for this item-ID, zero points, or specified partial points. We avoid awarding non-documented partial points.
- The two grading items E2-A3 and E2-A4 labeled with **GLOBAL** can be awarded for a criterion fulfilled in the specific subtask or anywhere else in the exam.

Överkurs, i.e. extra information

All solutions are given as a numerical result ± two statistical standard deviations, and all possible systematic errors are ignored. This is not required by the students.

E1 Probing the acoustic field

a) Circuit diagram

The transducers in the levitator act as transmitters and are arranged such that a standing acoustic field with pressure nodes and antinodes is created. In the nodes, the pressure variation is minimal due to destructive interference, whereas in the antinodes it oscillates most strongly. An additional transducer can be used as a receiver to probe this field and convert the pressure differences into an alternating voltage. This spatial variation can be visualized using light-emitting diodes.

Depending on the circuit configuration (see Fig. 3), the brightness differs significantly. With two antiparallel LEDs, the light output is strongest because one LED conducts in each half-cycle of the signal. If only a single LED is used, the light output is weak because only one half of the signal contributes and the LED is mainly operated in reverse-bias. Interestingly, if both leads of the LED are touched with the hand, a reasonable brightness can be observed. The human body introduces finite resistance, which reduces the reverse-bias problem and yields better brightness than in the completely open case, but still not as good as in the antiparallel case.

Överkurs, technical excursion

To probe the acoustic field, a piezoelectric transducer is used as a receiver. The incident sound wave induces a periodic mechanical deformation of the piezoelectric crystal, which displaces charges in the lattice and generates an alternating voltage across the electrodes. Thus, electrically, the transducer behaves like a capacitive AC source with position-dependent amplitude. The circuit rapidly reaches a steady state in which the net transferred charge is zero, meaning that only alternating currents flow. Grounding one node does not change this behavior; it merely sets a DC reference potential without improving the signal or light output. Using antiparallel LEDs provides the most efficient use of the available current, as each half-cycle has a forward conduction path. Replacing a diode with a resistor introduces a bidirectional current path that diverts current away from the LED, reducing brightness. The resistance value is critical: a short circuit bypasses the LED completely, while an open circuit forces charge balance through the LED alone, where the very small reverse current severely limits the forward current and thus light output. Touching both terminals with the hand introduces moderate resistance for this scenario, allowing sufficient current for visible light, though still significantly less than in the antiparallel configuration.

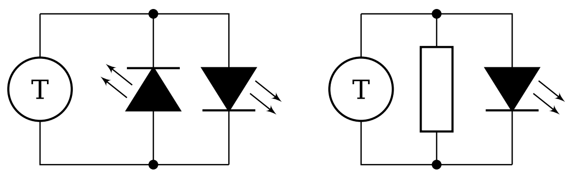

Figure 3: Circuit diagram of the alternating configuration (left) and LED with resistor configuration (right)

b) Distance between antinodes

The transducer head is vertically moved and the approximate height change is recorded using the ruler. For a vertical distance of $h_5 \approx 20\,\mathrm{mm}$, the LEDs light up five times, starting and ending with lit up LEDs. Thus, five anti-nodes have been measured. This yields the estimate:

$$\boxed{h = \frac{h_5}{5-1} \approx (5 \pm 2)\ \text{mm}} \quad (5)$$

Grading scheme E1

ID	Scoring Item	Points
E1-A1	Correct circuit diagram (does mot require correct symbol for transducer or diode, but alternate polarity should be clearly indicated).	**0.5 pts**
	Second circuit diagram (with resistor/hands)	0.3 pts
	Transducer and one diode only	0.2 pts
	In case diode symbol without polarity is used, independent of remaining circuit	0.1 pts
	Any circuit with a voltage source (power supply or levitator top rails):	0.0 pts
E1-B1	Transducer head is moved along the z-axis and vertical distance between at least two anti-nodes is measured.	**0.2 pts**
E1-B2	Result for h between adjacent antinodes is $\in [3, 8]$ mm.	**0.3 pts**
	Total points	**1.0 pts**

E2 Acoustic Levitator Properties

a) Frequency of the transducers

The distance between two adjacent nodes is $\lambda/2$, where λ is the wavelength of the acoustic wave. Indeed, when one moves $\lambda/2$ up, the phase of the sound wave from the transducers at the top part of the levitator decreases by π, while the phase of the waves from the transducers at the bottom part increases by π. This causes a total phase difference of 2π.

To determine the wavelength of the acoustic wave, one has to measure the location of the nodes. This can conveniently be achieved by levitating multiple beads of type (N) or (O). The advantage of the styrofoam beads (N) is that they are very light and hence deviate a little from the nodes (cf. task E3), while the advantage of the beads (O) is their regular shape. Levitating beads are shown in Fig. 4.

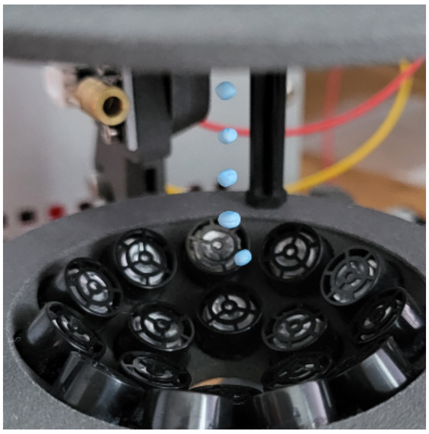

Figure 4: A photograph of the levitating styrofoam beads.

To magnify the image, illuminate the levitating beads with the red LED and project their shadows onto a screen, as shown in Figure 5. The magnification factor can, for example, be determined using a ruler in the following manner. One needs to place the ruler vertically in the center of the levitator, and determine the magnification factor from the distance between the projections of the vertical mm marks. Alternatively, the magnification factor can be deduced from the size of the shadow of beads (O) or (P), using that their real diameter is 2.0 mm. The described method for determining the magnification factor will be used in many of the subsequent tasks.

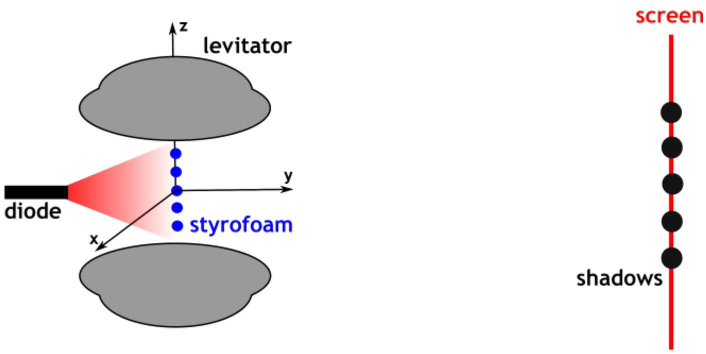

Figure 5: A sketch of the used measurement setup.

The obtained data are shown in Tab. 1, and Fig. 6 shows the distance from the 0:th bead as a function of the node number. The slope of the line is expected to be $\lambda/2$, and hence it is found that $\lambda \approx 10.12$ mm. Since $f = v/\lambda = \frac{340}{0.01012}$ Hz, this gives:

$$\boxed{f = (34 \pm 1)\ \text{kHz}} \quad (6)$$

Node	Dist. on screen [cm]	Real dist. [mm]
0	0.00 (reference)	0.00
1	7.85	5.16
2	15.45	10.17
3	23.15	15.23

Table 1: Example data showing the distance of the nodes from the 0th node for magnification factor $M = 15.2$.

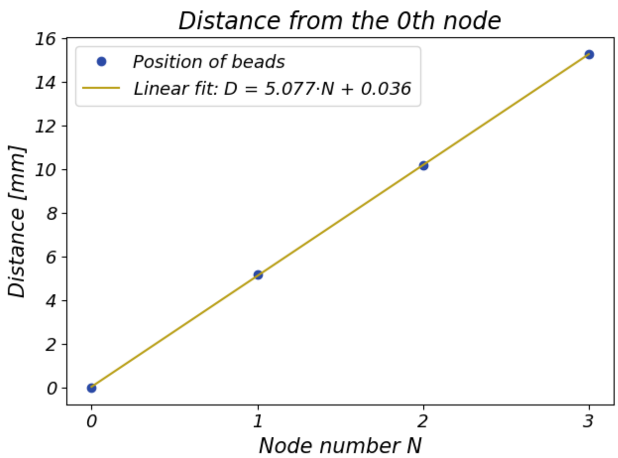

Figure 6: The distance from the 0:th bead in mm versus the number of the bead.

b) Mode X

When *Mode X* is turned on, a solid object placed in the acoustic field starts to move upward. This can be explained as follows. Let the wave produced by the lower half of the levitator, traveling along the positive z axis, be given by $p_l(z,t) = p_0 \sin(\frac{\omega}{v}z - \omega t)$. Similarly, let the wave produced by the upper half of the levitator, traveling along the negative z axis, be given by $p_u(z,t) = p_0 \sin\left(-\frac{\omega'}{v}z - \omega' t\right)$, where ω' is the slightly changed angular frequency. The resulting acoustic pressure is given by the sum of the two contributions:

$$\begin{aligned} p_{\text{tot}} &= p_l + p_u \\ &= p_0 \sin(\frac{\omega}{v}z - \omega t) + p_0 \sin\left(-\frac{\omega'}{v}z - \omega' t\right) \\ &= -2p_0 \sin\left(\frac{\omega - \omega'}{2v}z - \frac{\omega + \omega'}{2}t\right) \cos\left(\frac{\omega + \omega'}{2v}z + \frac{\omega' - \omega}{2}t\right), \end{aligned}$$

where we used the relation $\sin(a) + \sin(b) = 2\sin((a+b)/2) \cdot \cos((a-b)/2)$. The sine term in this expression is a wave with the very long wavelength $\tilde{\lambda} = \frac{2v}{\Delta f} \approx 1\,\text{km}$ and the angular frequency $\frac{\omega+\omega'}{2} \approx \omega$, so it can be approximated as $\sin(\omega t)$. The second term is almost a standing wave of wavelength $\frac{2v}{f+f'} \approx \frac{v}{f} = \lambda$, but with nodes moving with a small velocity $v = \frac{|\Delta\omega|}{2k}$, where $\Delta\omega = \omega' - \omega$. If $\Delta\omega > 0$, the nodes move in the direction of the negative z axis, and if $\Delta\omega < 0$, they move along the positive z axis. Using $\omega = 2\pi f$ and $k = 2\pi/\lambda$ we find that

$$v = \frac{\lambda|\Delta f|}{2}. \tag{7}$$

Since levitating beads move along the positive z axis when *Mode X* is on, it is deduced that $\Delta f < 0$. The value of $|\Delta f|$ can be found using Eq. 7, for example in the following way. First mark the position of the top part of a levitated bead's shadow on the screen, as it levitates in two different nodes (as widely separated as possible, to increase accuracy). Place the bead slightly below the lowest of these nodes. Then turn on *Mode X* and measure the time it takes for the top part of the bead's shadow to travel between the two marks on the screen. The frequency difference is found to be:

$$v = \frac{\lambda|\Delta f|}{2} \Rightarrow |\Delta f| = \frac{2}{\lambda}\frac{N\frac{\lambda}{2}}{t} = \frac{N}{t}, \tag{8}$$

where $v = s/t = (N\lambda/2)/t$ was used. The results of a series of measurements for the travel time over a distance of four nodes are shown in Table 2. The average of 5.96 seconds on a distance of four nodes gives us $|\Delta f| = 4/5.96 \approx 0.67\,\text{Hz}$.

Alternatively, one can describe the phenomenon by the Doppler effect in the system of reference moving with speed u according to Eq. 9, where the nodes do not move relative to the ball. Therefore, the frequency from the upper and lower half would be measured to be the same in this reference frame.

$$\left(\frac{v+u}{v}\right)(f + \Delta f) = \left(\frac{v-u}{v}\right)f \tag{9}$$

Using that the ball is moving upward $u > 0$ and the approximation $u << v$ one arrives at the relation in Eq. 10, where u is the speed of the styrofoam bead and v the speed of sound.

$$\Delta f = -\frac{2u}{v}f \tag{10}$$

Measurement	Time [s]
1	5.84
2	6.00
3	5.94
4	5.96
5	6.06
Average	**5.96**

Table 2: One measurement series for the time it takes for a bead to travel a distance of four nodes.

$$\boxed{\Delta f = (-0.67 \pm 0.02)\ \text{Hz}} \tag{11}$$

Grading scheme E2

ID	Scoring Item	Points
E2-A1	Distance between nodes: $\lambda/2$	**0.2 pts**
E2-A2	Distance over three or four nodes measured OR PP/glass beads used in two nodes and argued for this choice using regular shape.	**0.3 pts**
E2-A3	**GLOBAL: Magnification calibration method**. Method statement, e.g., bead or ruler optically magnified, full points. Linearization *and* determining LED emission point (≈ 5 mm inside) awarded full points if done precisely.	**0.4 pts**
E2-A4	**GLOBAL: Magnifying setup** Sketch of an optical magnification setup.	**0.3 pts**
E2-A5	Measured raw vertical distance in magnified image (on ruler) ≥ 20 cm.	**0.4 pts**
	For vertical distance ≥ 10 cm	0.2 pts
E2-A6	Result for $f \in [30, 40]$ kHz (this interval should be rescaled accordingly if the answer in E2-A1 is wrong by a numerical factor).	**0.4 pts**
E2-B1	Correct theoretical derivation of formula $v = -\frac{\lambda \cdot \Delta f}{2}$ (the derivation may assume that both plane waves have the same wavenumber $k = \frac{\omega}{v}$).	**0.5 pts**
	If factor $\frac{1}{2}$ missing.	0.3 pts
E2-B2	Phenomenon described, e.g., beads moving on the screen OR measuring the blinking frequency of a fixed LED with *Mode X* turned on.	**0.3 pts**
E2-B3	At least 5 time measurements with beads OR averaged over at least 10 blinkings of LED.	**0.5 pts**
	for $x < 5$ measurements, per measurement OR for each pair of blinkings, up to 10 blinkings	0.1 pts
E2-B4	Using the formula $v = s/t$	**0.1 pts**
E2-B5	Correct value $\lvert\Delta f\rvert \in [0.66, 0.68]$ Hz (should be rescaled accordingly if answer in E2-A1 is wrong by numerical factor)	**0.4 pts**
	$\lvert\Delta f\rvert \in [0.64, 0.66[\cup]0.68, 0.70]$ Hz	0.2 pts
E2-B6	Correct sign of Δf (negative)	**0.2 pts**
	Total points	**4.0 pts**

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- The operating frequency of the transducers is $f = 40$ kHz, which corresponds to a wavelength of $\lambda = 8.5$ mm. The systematic error in the frequency measurement can be explained by the fact that the acoustic waves generated by the two halves of the levitator are not plane. See Fig. 7, for the simulated acoustic pressure field generated by a levitator of a design similar to the one used in the task.

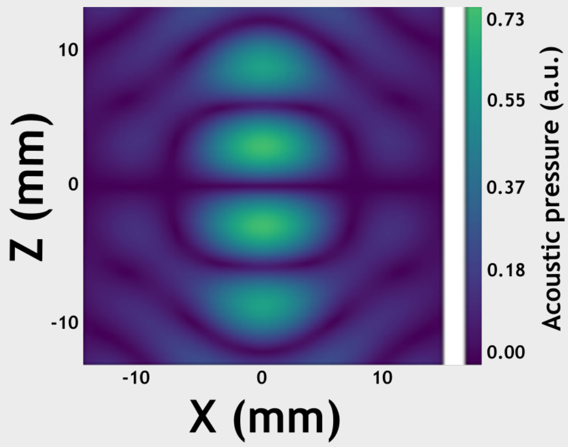

Figure 7: Acoustic field of MK1 setup. Figure from j.jsamd.2024.100720

- You may have noticed that the bead does not rise quite smoothly when *Mode X* is on, but rather moves in short "jumps". The reason is that, in reality, *Mode X* is implemented by changing the phase difference between the two halves in finite (small) steps. Since each phase difference change is very small, this amounts approximately to a continuously increasing phase difference, which is mathematically equivalent to a frequency offset.

E3 Determining the density

The unknown material is polypropylene plastic (PP).

Points can only be awarded for one of the methods A and B below. If both methods are attempted, the method yielding the highest total points is used.

Method A (deviation from node)

In this task, the same magnification approach as in Task E2 shall be used. However, the magnification should be made larger (since only one bead needs to be visible). A magnification of at least $M > 30$ is obtained.

Start by finding the location of the node. A small piece of styrofoam is levitated approximately in the center of the node and marked on the screen.

According to the hint, the acoustic radiation force can be approximated by a spring force $\vec{F} = -f(R)P^2\Delta\vec{z}$. The equation of force equilibrium for the levitated sphere is:

$$-f(R)P^2\Delta z = mg = \rho V g, \quad (12)$$

where m is the mass of the bead, ρ is the its density and V is its volume. Using the assumption that the acoustic pressure is proportional to the voltage $P \propto U$ it is clear that $|\Delta z| \propto U^{-2}$.

Since the glass and plastic beads have the same shape and size, the slope k of the obtained line for Δz vs. $1/U^2$ is proportional to the density of the levitated object. Thus, it follows that

$$\rho_{\text{PP}} = \rho_{\text{glass}} k_{\text{PP}}/k_{\text{glass}} \quad (13)$$

The glass is levitated at the maximum voltage and the offset from the node is measured (on the screen, the image is magnified) for different voltages. At least five measurement points are required to get accurate results. The measurements are repeated for the plastic bead. From the slope of the Δz vs. U^{-2}-graphs in Fig. 8 the densities are calculated according to Eq. 13.

$$\boxed{\rho_{PP} = (800 \pm 60)\ \text{kg}\,\text{m}^{-3}} \quad (14)$$

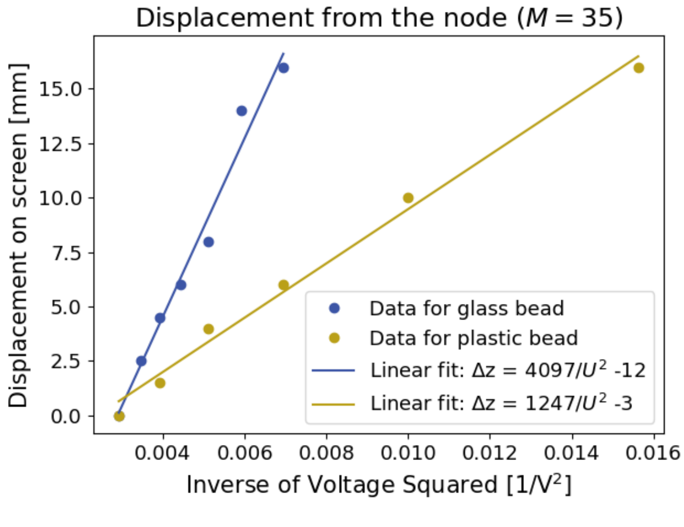

Figure 8: Density plot for plastic and glass bead with a magnification factor $M = 35$.

Grading scheme E3: method A (deviation from node)

ID	Scoring Item	Points
E3-A1	Correct force balance $-f(R)P^2\Delta z = mg = \rho V g$ (one version enough, not grading sign).	**0.3 pts**
E3-A2	$\rho_{\text{glass}} k_{\text{PP}}/k_{\text{glass}}$ gives the density of PP (or if pairwise measurements for constant voltage, ratio of displacements).	**0.5 pts**
E3-A3	Magnification factor ≥ 30 or if diode the closest possible to levitator distance from node to screen ≥ 66 cm	**0.4 pts**
	Magnification factor ≥ 20 or if diode the closest possible to levitator distance from node to screen ≥ 43 cm	0.2 pts
E3-A4	At least 4 voltage and displacement measurements for glass bead (the voltage may be the same for all measurements)	**0.5 pts**
	For 3 measurements	0.3 pts
	For 2 measurements	0.2 pts
	For 1 measurements	0.1 pts
E3-A5	At least 4 voltage for plastic bead (the voltage may be the same for all measurements)	**0.5 pts**
	For 3 measurements	0.3 pts
	For 2 measurements	0.2 pts
	For 1 measurements	0.1 pts
E3-A6	Graphical evaluation (slope) for both materials OR linear regression on calculator. Pointwise average yields no points, unless clearly stated that the exact location of the node is determined (e.g. by levitating a styrofoam bead at a high voltage in the same node, marking the position on the screen).	**0.4 pts**
E3-A7	Density within the interval $\rho_{PP} \in [750, 1150]$ kg/m^3	**0.4 pts**
	Total points	**3.0 pts**

Method B (drop voltage)

An alternative method to compare the densities of the beads of type (O) and (P) is to analyze the drop voltage, i.e. the smallest voltage at which the beads can be levitated. Indeed, by levitating at different voltages, one notices that denser objects require a higher voltage to be levitated.

The approach is based on the observation that a levitating spherical bead of a fixed size will fall precisely when its deviation from the node, Δz, reaches a certain value $\Delta z_{\max}$, independent of the bead's density. At this point, the acoustic force reaches a maximum, so a further increase of Δz only results in a lower restoring force. If one considers the restoring acoustic force as a spring this corresponds, in some sense, to the spring breaking.

According to the theory presented in the task, the acoustic force on a bead of a fixed shape at the deviation $\Delta z_{\max}$ is given by $F = KU^2$, where K is a constant. Hence, the drop voltage $U_{\min}$ satisfies

$$KU_{\min}^2 = \rho g V,$$

where V is the volume of the bead. Thus, we conclude that

$$\rho_{\text{PP}} = \rho_{\text{glass}} \frac{U_{\text{PP}}^2}{U_{\text{glass}}^2} \tag{15}$$

The plastic and glass bead are levitated in the central node. The voltage is slowly decreased and the drop voltage is recorded. At least three measurements should be done for the drop voltage of each bead type. The average drop voltage is used to calculate the density of the plastic bead according to Eq. 15. Alternatively, it could be argued that the minimum of the recorded drop voltage values should be used. Indeed, due to the shaking of the bead, it may fall at a voltage higher than $U_{\min}$, and therefore $U_{\min}$ should be best approximated by the minimum voltage of the recorded values.

Some example data is shown in Tab. 3 and the density result based on the example data is:

$$\boxed{\rho_{\text{PP}} = (1010 \pm 20)\ \text{kg}\,\text{m}^{-3}} \tag{16}$$

	U_{glass} [V]	U_{PP} [V]
	11.1	7.1
	11.1	7.1
	11.1	7.0
	11.0	7.1
	11.1	7.0
Average:	**11.08**	**7.04**

Table 3: Example data showing drop voltage for glass and plastic bead.

Grading scheme E3: method B (drop voltage)

ID	Scoring Item	Points
E3-B1	Idea that the drop voltage is influenced by the density of the beads.	**0.5 pts**
E3-B2	Idea that there is a maximum displacement for which the beads fall.	**0.5 pts**
E3-B3	Correct force balance $KU^2 = \rho g V$ or $KU^2 = mg$	**0.3 pts**
E3-B4	Idea that $\rho_{\text{glass}} U_{\text{PP}}^2 / U_{\text{glass}}^2$ gives the density ρ_{PP}	**0.3 pts**
E3-B5	At least 3 voltage measurements for glass bead	**0.5 pts**
	For 2 voltage measurements	0.3 pts
	For 1 voltage measurements	0.1 pts
E3-B6	At least 3 voltage measurements for plastic bead	**0.5 pts**
	For 2 voltage measurements	0.3 pts
	For 1 voltage measurements	0.1 pts
E3-B7	Correct density within range $\rho \in [900, 1100]$ kg/m^3	**0.4 pts**
	Total points	**3.0 pts**

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- The distance between two adjacent nodes is $\lambda/2$ (NB! this is not the wavelength of the acoustic force in blue, which is $\lambda' = \lambda/2$). See Fig. 9 for a sketch of acoustic pressure force. As in described in method B, there is a point where an increase in the displacement does not give an increase in the acoustic force. This corresponds to a spring becoming permanently deformed. This happens at $\Delta z \approx \lambda/8$.

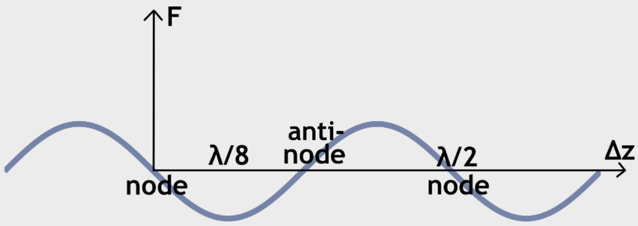

Figure 9: Periodic acoustic force.

- There is a comparatively simple formula describing the acoustic force on a spherical bead whose radius is much smaller than the wavelength of the ultrasound:

$$F = 4\pi\Phi_0 k R^3 E_{\text{ac}} \sin(2k\Delta z).$$

Here Φ_0 is a dimensionless contrast factor depending on the compressibilities and densities of the bead and the ambient air, $k = \frac{2\pi}{\lambda}$ is the wavenumber, and $E_{\text{ac}} \propto P^2$ is the acoustic energy density. In this task, however, it is not accurate to assume that the droplets are much smaller than the wavelength, and the dependence is more complicated.

E4 Evaporation

The liquid I in question is isopropanol (IPA).

a) Evaporation constant

The given evaporation model states that the squared diameter of the droplet decreases linearly with time. To measure evaporation, one has to calibrate the setup in a similar manner as in E2. Since the given equation holds for spherical droplets, one has to convert the ellipsoidal shape to a sphere, according to $abc = r^3$, i.e. $r = (a^2c)^{1/3}$, where we used $a = b$ (by symmetry). This can be written in terms of the diameter D as $D = 2(a^2c)^{1/3}$ where a is the semi-major axis and c is the semi-minor axis of the ellipsoidal droplet.

To estimate γ, measure the major and minor axes and wait in between measurements. The task asks to determine γ when $D \gtrsim 1.5$ mm ($D^2 \gtrsim 2.25$ mm^2) so data are collected when the droplet is both larger and slightly smaller than this size (for task E4b). One has to convert the results first to the observed diameter D with $D = 2(a^2c)^{1/3}$ and then to the real diameter $\mathcal{D}$ with $\mathcal{D} = D/M$, where M is the magnification factor. An example dataset is shown in Tab. 4 and visualized in Fig. 10.

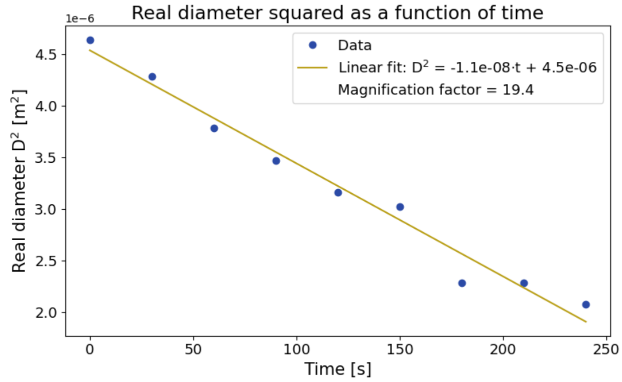

Figure 10: Sizes of the real diameter squared, see Tab. 4, plotted as a function of time.

The value of γ is then read as the slope of the curve which gives us the result:

$$\boxed{\gamma = (9.2 \pm 5.0) \cdot 10^{-9}\ \text{m}^2/\text{s}} \tag{17}$$

b) Evaporation time

When one waits long enough one can observe that the evaporation rate slows down considerably when a certain size is reached. In Figure 11 one sees that this happens after 600 seconds at about $D = 1$ mm. We see that a prediction based on the first slope would have led to a drastic underestimation of the evaporation time.

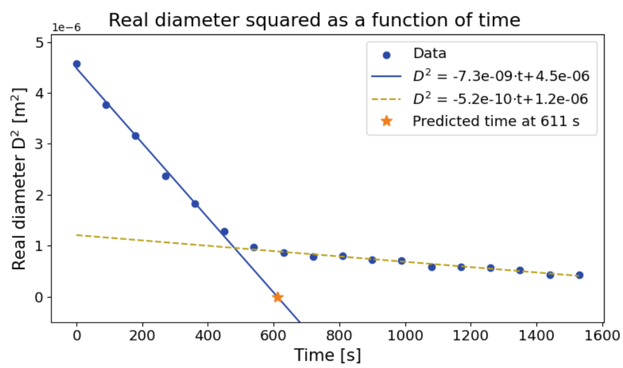

Figure 11: Sizes of the real diameter squared plotted as a function of time. The used magnification factor was 30. We see that the evaporation levels off after a certain size and that a prediction based on the original slope would have been significantly off (predicted time, marked with a star sign)

Time [s]	Major [mm]	Minor [mm]	Diameter [mm]	Real [m^2]
0	47	33	4.2e+01	4.6e-06
30	45	32	4.0e+01	4.3e-06
60	41	32	3.8e+01	3.8e-06
90	39	31	3.6e+01	3.5e-06
120	37	30	3.5e+01	3.2e-06
150	37	28	3.4e+01	3.0e-06
180	30	28	2.9e+01	2.3e-06
210	30	28	2.9e+01	2.3e-06
240	29	26	2.8e+01	2.1e-06

Table 4: Example data showing time for the measurement, on screen major axis, on screen minor axis, diameter of the corresponding sphere on the screen and the real, actual diameter of the droplet squared. The scale factor for this measurement was 20.

Grading scheme E4

ID	Scoring Item	Points
E4-A1	Both major axis and minor axis measured.	**0.3 pts**
E4-A2	At least 9 data points for $D > 1.3$ mm, with at least $20\,s$ between the measurements (if initial D determined from volume measured using the inaccurate scale of the syringe, no points).	**0.7 pts**
	at least 7 data points	0.5 pts
	at least 5 data points	0.3 pts
	at least 3 data points	0.1 pts
E4-A3	Measuring the droplet for a total time of at least $4\,\text{min} = 240\,\text{s}$ above $D > 1.3$ mm.	**0.2 pts**
E4-A4	Diameter of corresponding sphere calculated correctly $D = 2(a^2c)^{1/3}$.	**0.2 pts**
E4-A5	Magnification $M \geq 20$	**0.2 pts**
	Magnification $M \geq 15$	0.1 pts
E4-A6	Correct linearization (integrated version of Eq. 3 D^2, against t or idea to plot D^2 against t).	**0.2 pts**
E4-A7	Determining γ as the slope, graphically or by using a linear regression on calculator, of recorded data D^2 vs. t.	**0.2 pts**
E4-A8	Value of $\gamma \in [2.5, 13.4] \cdot 10^{-9}$ m^2/s.	**0.5 pts**
E4-B1	Observation that the droplet's evaporation slows down when a certain size is reached.	**0.3 pts**
E4-B2	Statement that the result in part a) would have led to an underestimation of the evaporation time.	**0.2 pts**
	Total points	**3.0 pts**

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- The relation $d(D^2)/dt = -\gamma$ is known as the D-squared law, and it is typically assumed to hold for small spherical droplets. In this problem we have applied it explicitly to non-spherical droplets by converting the axes to an equivalent diameter and found out that it still holds.
- Evaporation is a phenomenon that is affected by a lot of variables. The D-square law is a simple but a useful tool to describe the phenomenon. Its downside is that it does not take into account differences caused by temperature, humidity, air pressure, shape of the droplet, draft or turbulence by the acoustic field. We have tried to take into account these effects by measuring in different conditions which has lead to a large interval of accepted solutions.
- Observe the strong change in the value of γ when we finally get to aspect ratio 1, *i.e.*, the ratio between the major axis and the minor axis, equaling to 1. This shows that the used model is too simple to explain the full range of the phenomena. The moral of the problem can be summarized by the words of George Box: *"All models are wrong, but some are useful."*

E5 Determining the Surface Tension

a) Surface tension ranking

One can quickly determine which of the liquids has a lower surface tension in a number of ways. One idea is to just place two similarly big droplets of the liquids on the table surface and visually notice that

U_{II} [V]	U_I [V]	σ [mN/m]
16.3	9.6	25.3
17.8	10.3	24.4
14.3	8.5	25.8
16.4	9.7	25.5
15.6	9.0	24.3

the contact angle for liquid I is much lower. Another approach is to levitate similarly big droplets in the levitator and deform them or make them explode and note that this happens at significantly lower voltages for liquid I. Alternatively, just solving Task b) quantitatively also gives the right qualitative answer.

b) Unknown surface tension

The shape of a droplet only depends on its volume V and the ratio P^2/σ but in an unknown manner. To solve the task, we can measure pairs of droplets (one of each liquid) with equal volume $V_{\mathrm{I}} = V_{\mathrm{II}} = \mathrm{const.}$ and apply a voltage such that their shape matches. This implies $P^2/\sigma \propto U^2/\sigma = \mathrm{const.}$
Experimentally, the already established magnification setup is used and we start levitating a droplet of liquid II and deform it at voltage U_{II} such that it is clearly non-spherical. For a precise measurement of σ, it is necessary to significantly deform the droplet because otherwise the contribution of P^2/σ to the overall shape becomes too small. We mark the shape of the droplet on the screen. Next, we remove the droplet from the levitator and now levitate a droplet of liquid I with the same volume. We tune the voltage of the levitator such that the shape matches our markings made before and write down this voltage U_{I} as well. To get a droplet of the same volume, it is a good strategy to levitate a slightly larger droplet at first and then wait a bit and use the (previously explored) evaporation to fine-tune the volume to match the shape exactly.

Finally, the surface tension can be determined via the known surface tension σ_{II} as:

$$\sigma_{\mathrm{I}} = \frac{U_{\mathrm{I}}^2}{U_{\mathrm{II}}^2}\sigma_{\mathrm{II}} \quad (18)$$

and we repeat the measurement 5 times to increase our accuracy.

$$\boxed{\sigma_{\mathrm{I}} = (25.1 \pm 1.3)\ \mathrm{mN/m}} \quad (19)$$

Grading scheme E5

ID	Scoring Item	Points
E5-A1	Effective idea to compare the surface tension.	**0.2 pts**
E5-A2	Correct conclusion $\sigma_{\mathrm{I}} < \sigma_{\mathrm{II}}$.	**0.3 pts**
E5-B1	Idea to use symmetry *i.e.*, V = const. **and** same droplet shapes (pairwise at least). If contact angle approach used without magnification ≤ 10, no poits are awarded for the entire task E5b	**0.5 pts**
E5-B2	Obtaining or using P^2/σ = const.	**0.2 pts**
E5-B3	Repeating the correct measurement procedure (evidenced by pairs of voltages) for 5 repetitions.	**0.3 pts**
	For at least 3 repetitions	0.2 pts
	For at least 1 repetitions	0.1 pts
E5-B4	Result for σ_{I} within $[21, 29]$ mN/m	**1.0 pts**
	$\sigma_{\mathrm{I}} \in [19, 21[\cup]29, 31]$ mN/m	0.5 pts
	$\sigma_{\mathrm{I}} \in [16, 19[\cup]31, 34]$mN/m	0.2 pts
	Total points	**2.5 pts**

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- Liquid I is IPA (isopropanol), which has a surface tension of 21.8 mN/m according to the literature. The method overestimates the surface tension slightly but is reasonably accurate considering the measurement technique.

E6 Explosion of Droplets

a) Fitting parameters

Measure the explosion voltage for droplets of different equivalent diameters D. By Equation (4), a plot of U_{max}^2 against $\frac{1}{D}$ is expected to yield a straight line, and the constants α and β are determined as the slope and y-intercept of this line, respectively. The obtained data is presented in Table 5, and the corresponding plot is shown in Figure 12. The obtained values for α and β are:

$$\boxed{\alpha = (569 \pm 54)\ \mathrm{V^2{\cdot}mm}, \quad \beta = (-36 \pm 26)\ \mathrm{V^2}}$$

$2a$ [mm]	$2b$ [mm]	D [mm]	$1/D$ [mm^{-1}]	U_{max} [V]	U^2_{max} [V^2]
73	63	1.70	0.590	17.7	313.3
87	65	1.93	0.519	15.9	252.8
78	68	1.82	0.550	16.7	278.9
140	71	2.72	0.367	13.2	174.2
122	62	2.38	0.421	14.1	198.8
99	59	2.03	0.492	15.6	243.4
102	70	2.19	0.456	14.6	213.2
80	52	1.69	0.592	17.5	306.3
142	66	2.68	0.373	13.1	171.6
117	72	2.43	0.412	13.8	190.4
113	67	2.32	0.432	14.6	213.2
87	52	1.79	0.559	16.7	278.9
165	87	3.25	0.308	12.5	156.3

Table 5: Data corresponding to the measurements of exploding droplets. The axes of the droplet are measured in mm on the screen. The magnification is 41.

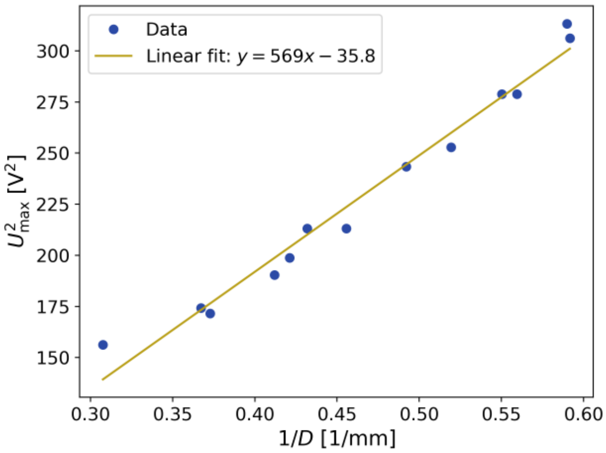

Figure 12: Plot of U^2_{max} against $1/D$.

To determine the equivalent diameter D of a levitated droplet, use the technique described in Task E4. Note that the approximation of a droplet as an ellipsoid is most accurate when the droplet is round, so the diameter of the droplet should be measured at a comparatively low voltage. To reduce the evaporation of the droplet between its explosion and the diameter measurement, one needs to quickly increase the voltage to a moderately high value, guided by the shape of the droplet. Then change the voltage by $0.1\,\mathrm{V}$ at a time until the droplet explodes. Often, when a large droplet explodes, the remains form a smaller droplet, which can be used for the next measurement, to conveniently obtain a large variation of D values.

b) Maximum diameter to levitate

In this task, two effects preventing the droplet from levitating are considered:

1. If the applied voltage is too high, the droplet explodes, as analysed in the previous subtask. This yields an upper bound $U_{max} = U_{max}(D)$ for the voltage, decreasing with D.

2. If the applied voltage is too low, the droplet falls. This yields a lower bound $U_{min} = U_{min}(D)$, increasing with D.

An estimate of the equivalent diameter D_{max} of the largest droplet that can be levitated is obtained as the intersection of the two curves corresponding to the above restrictions. The explosion voltage was analyzed in the previous subtask, so what remains to understand is the dependence between U_{min} and D. Note that small droplets can be created by exploding larger droplets, as in the previous subtask. The recorded measurement values are presented in Table 6, and U_{min} and U_{max} are plotted against D in Figure 13.

$2a$ [mm]	$2b$ [mm]	D [mm]	U_{min} [V]
99	76	2.61	8.7
87	69	2.32	8.4
124	80	3.08	10.0
121	82	3.07	9.9
63	58	1.77	7.7
74	61	1.99	8.0
54	45	1.47	7.7
86	65	2.26	8.9
125	83	3.14	10.1
124	84	3.14	10.0
114	85	2.97	9.6
80	60	2.09	8.0
111	77	2.82	9.3
83	62	2.17	8.1
148	81	3.48	11.0
134	76	3.19	10.1
69	54	1.83	7.7
58	49	1.58	7.2
105	69	2.63	8.9
89	62	2.27	8.2

Table 6: Data corresponding to the measurements of drop voltages. The axes of the droplet are measured in mm on the screen. The magnification is 35.

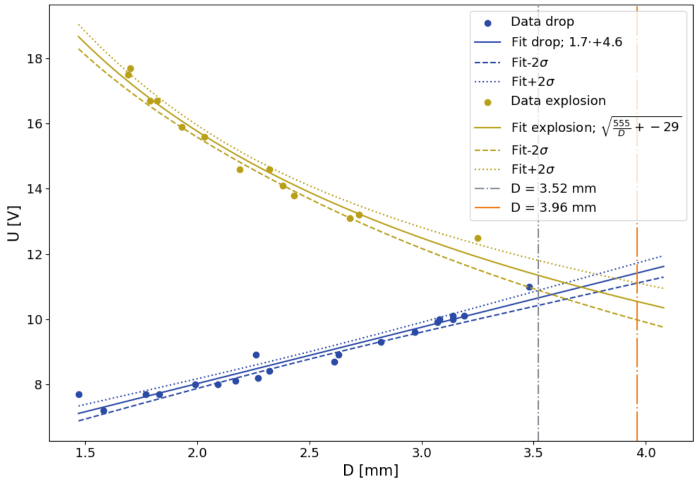

Figure 13: Plots of U_{min} and U_{max} against D together with fitted curves.

As seen in the figure, the fall voltage is described fairly well by an affine linear function, *i.e.*, $U_{\text{min}} = aD + b$. Hence, by Equation (4), D_{max} is obtained by solving

$$\sqrt{\frac{\alpha}{D} + \beta} = aD + b.$$

This equation can, for example, be solved graphically or by using the bisection method. It is found that

$$\boxed{D_{\text{max}} \approx (3.7^{+0.3}_{-0.2})\ \text{mm}}$$

which is the desired result.

Grading scheme E6

ID	Scoring Item	Points
E6-A1	Correct linearization (*i.e.*, the idea that a plot of U^2_{max} against $1/D$ has slope α and y-intercept β). A table with U^2_{max} and $1/D$ data suffices as evidence to get these points.	**0.4 pts**
E6-A2	At least 10 data points measured for the explosion curve (if volume measured using the inaccurate scale of the syringe, no points shall be awarded here).	**0.5 pts**
	At least 8 data points	0.4 pts
	At least 6 data points	0.3 pts
	At least 4 data points	0.2 pts
	At least 2 data points	0.1 pts
E6-A3	Values of D span at least $1\,\text{mm}$.	**0.3 pts**
E6-A4	Correct value: $\alpha \in [500, 700]\ \text{V}^2 \cdot \text{mm}$.	**0.4 pts**
	$\alpha \in [400, 500[\cup]700, 800]\ \text{V}^2 \cdot \text{mm}$	0.2 pts
E6-A5	Correct value: $\beta \in [-60, 60]\ \text{V}^2$.	**0.4 pts**
	$\beta \in [-80, -60[\cup]60, 80]\ \text{V}^2$	0.2 pts
E6-B1	Idea that the maximum droplet size is found at the intersection of the fall curve and the explosion curve.	**0.5 pts**
E6-B2	At least 10 data points measured for the drop voltage (if volume measured using the inaccurate scale of the syringe, no points shall be awarded here).	**0.5 pts**
	At least 8 data points	0.4 pts
	At least 6 data points	0.3 pts
	At least 4 data points	0.2 pts
	At least 2 data points	0.1 pts
E6-B3	Values of D span at least $1\,\text{mm}$.	**0.2 pts**
E6-B4	At least one point with $D > 3\,\text{mm}$ measured.	**0.1 pts**
E6-B5	Reasonable method for determining intersection of U_{min} and U_{max} (for example a graphical solution or the bisection method).	**0.3 pts**
E6-B6	Maximum diameter $D_{\text{max}} \in [3.5, 6]\ \text{mm}$	**0.4 pts**
	Total points	**4.0 pts**

E7 The Mysterious Line

The horizontal line is formed when the droplet is quite flat, typically around $0.5\ V$ before reaching the explosion voltage. For imperfect alignment (adjustable via the hex-screws) the line might also appear below or above the droplet. In Fig. 14, you can see an image of the phenomenon.

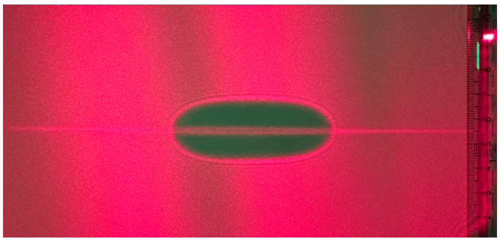

Figure 14: A photo of the line phenomenon.

To break the optical effect of the droplet down, it can be modeled as the superposition of two plano-convex lenses with liquid in between them in the vertical cross-section and a round (cylindrical) lens in the horizontal cross section.

The line emerges when incident rays from the diode propagating in the *vertical* plane are focused to a point on the screen, as shown in the simulation in Fig. 17. Since the dimensions of the droplet are much smaller than all other relevant distances, both the incident and emergent rays are approximately parallel to the optic axis. Hence, the situation in Figure 17 is obtained when the length of the droplet is

$$L \approx 2f, \tag{20}$$

where f is the focal length of the lenses in the vertical direction. The ray propagation inside the droplet can be seen in Figure 15.

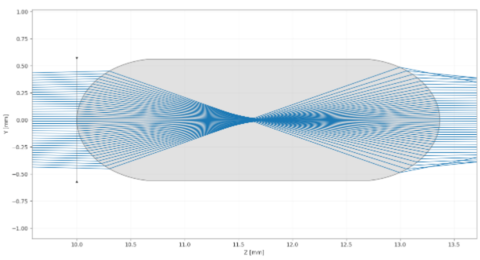

Figure 15: A zoomed-in part of Figure 17.

This observation is used to determine the index of refraction n of the analyzed liquid. Note that the focal length f is given by $f = R\frac{n}{n-1}$, where R is the radius of curvature of the two lenses in the vertical plane (near the tip of the droplet). Hence, by Eq. (20), the index of refraction of the liquid can be determined as

$$n = \frac{L}{L-2R}. \tag{21}$$

One way of determining the radius of curvature R of one of the lenses is the following. First sketch the contour of the levitating droplet when the line is visible. Then draw two lines perpendicular to the contour near the vertical center of the droplet. The distance from the outermost point of the droplet to the point of intersection of these two lines is R. This is shown in Fig. 16. This measurement is repeated

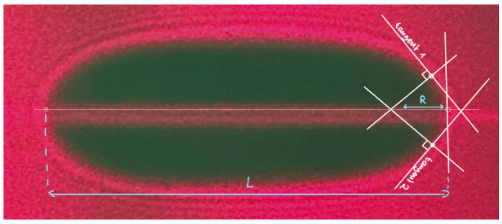

Figure 16: Geometrical construction to determine R and L for a given droplet using the ruler.

a number of times to increase the precision of the result. The final result is then the average of the single measurements.

$$\boxed{n = 1.34 \pm 0.1} \tag{22}$$

L [cm]	R [cm]	n
15.2	2.2	1.41
21.2	2.45	1.30
17.5	2.2	1.34
17.5	2.0	1.30
19.7	2.7	1.38

Table 7: Experimental data for E7

Due to the refraction in the *horizontal* direction, a line is formed instead of a point, as shown in the simulation in Fig. 18.

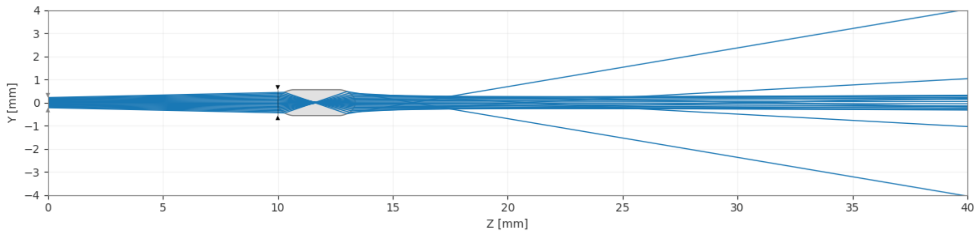

Figure 17: A simulation of the droplet focusing incident light beams from the diode to a point on the screen.

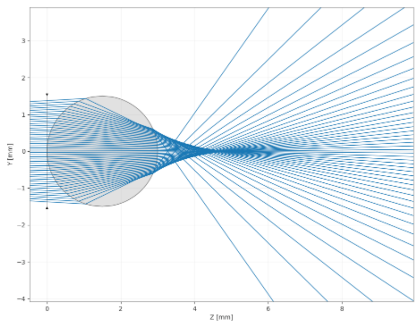

Figure 18: A simulation of the rays in the horizontal cross-section through the center of the droplet.

Grading scheme E7

ID	Scoring Item	Points
E7-1	Vertical contribution: Idea that line is formed when $L \approx 2f$, together with sketch as in Fig. 15. Ray propagation physically correct. Symmetry stated by text or labeling similar angles.	**0.8 pts**
E7-2	Horizontal contribution: Explain why line is formed instead of point, Fig. 18. Labeled.	**0.4 pts**
	Idea to split into vertical and horisontal contribution	0.2 pts
E7-3	Correct relation $L = 2Rn/(n-1)$ OR using Snell's law and clearly indicating relevant angles (both only if E7-1 was awarded).	**0.4 pts**
	Incorrect relation $L = 2R/(n-1)$ - using the lensmaker's formula, forgetting second half of lens is liquid (if E7-1 awarded)	0.1 pts
E7-4	Physically possible method for estimating R, reasonable value of R obtained ($R \in [0.1, 0.8]$ mm).	**0.4 pts**
E7-5	At least 3 measurements (measurements consist of (R, L) pairs OR angles for Snell's law).	**0.3 pts**
E7-5	At least 2 measurements.	0.2 pts
E7-5	At least 1 measurement.	0.1 pts
E7-6	Refractive index $n \in [1.2, 1.6]$.	**0.2 pts**
	Total points	**2.5 pts**

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- The unknown red liquid is water with some red food dye.
- The explanation given above suggests that the line should be visible only when the length L of the droplet and the radius of curvature R match perfectly. Hence, one could expect it to be very hard to observe the line phenomenon in practice. However, the conditions are somewhat relaxed due to aberrations of the two lenses. The incident rays are not focused into a single point, but rather into a short interval, and this allows for some flexibility in the constraints. The phenomenon is similar to an astigmatic lens.
- When light passes through the droplet and forms a shadow, Fraunhofer diffraction causes the light to spread and interfere, producing bright and dark fringes around the shadow rather than a perfectly sharp edge. The effect becomes more noticeable when the droplet size is small or when the observation screen is far from the droplet. This makes the apparent edge position depend on the measurement setup and potentially introduces systematic errors in the estimated droplet size.

E8 Experimental Techniques

Grading scheme E8

ID	Scoring Item	Points
E8-1	The box of the experimental setup is torn into pieces	-0.2 pts
E8-2	A LED is broken/blown up. Per LED broken.	-0.1 pts
E8-3	Some equipment is missing.	-0.1 pts
E8-4	The electrical control box is broken	-0.1 pts
	Total points	**-0.5 pts**