# Adiabatic Invariants

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## Contents



Adiabatic invariants are a powerful tool for solving certain problems in classical mechanics. Unfortunately, they are rarely taught outside of advanced undergraduate mechanics courses. In these notes, I will explain adiabatic invariants using the idea behind problem 3 of EuPhO 2025. I will assume you've studied that problem, and also that you're familiar with the Hamiltonian formulation of classical mechanics for systems with one degree of freedom.

## §1 Introduction

Adiabatic invariants appear in problems involving an oscillating mechanical system with a parameter that *varies slowly with time*. As an example of such a system, consider a pendulum consisting of a bob, of mass $m$, hanging from some massless string that passes through a hole in the ceiling (see figure 1(a)). We can vary a parameter by pulling the string up through the hole, changing the pendulum's length $\ell(t)$ (see figure 1(b)). This variation is 'slow' if the characteristic time, $T_\ell$, over which the length changes is much greater than the pendulum period, $T(t)$, at any time $t$ during the variation.

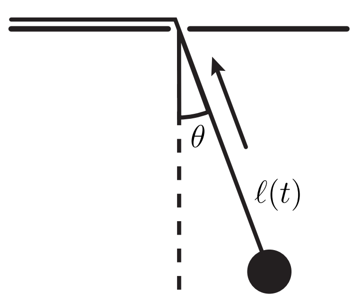


FIGURE 1: Pendulum whose length can vary with time

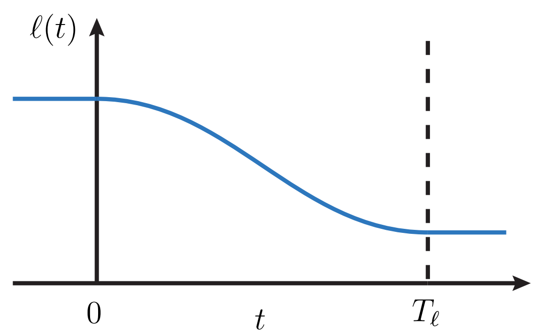


FIGURE 2: Pendulum length vs. time

By varying a parameter, we usually change the energy of the system. In the pendulum example, the energy of the bob is not conserved because we do work on the bob by pulling the string upwards. A natural question to ask is then: by how much does the energy change between the moment when we start pulling the string and the moment when we stop? This exact problem was posed by Hendrik Lorentz (after whom the Lorentz force and Lorentz transformations are named) to the other physicists attending the Solvay conference in 1911 (see figure 3). Einstein gave the

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answer immediately (see Einstein (1994)): the energy changes such that the ratio of the energy to the frequency of the pendulum remains constant[‡]. Below, we will see that the constant ratio of energy to frequency in this problem is an example of an adiabatic invariant.

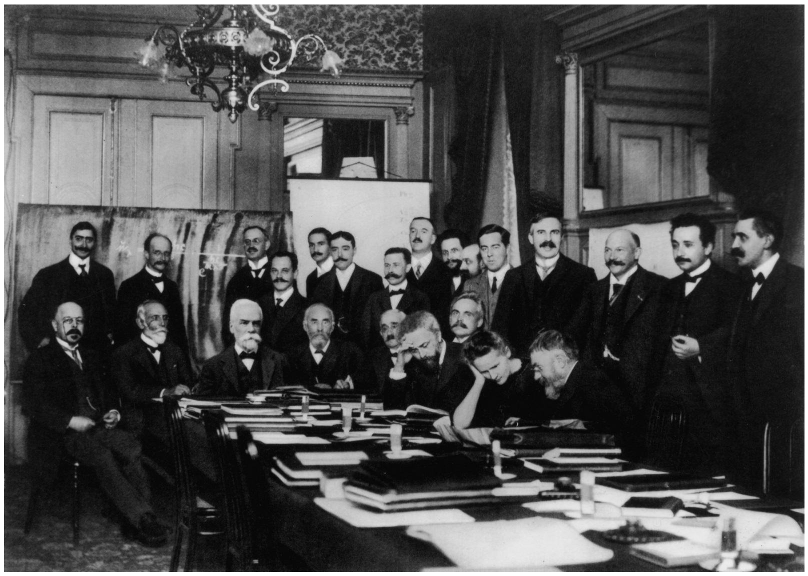

FIGURE 3: Photo taken at the first Solvay conference, 1911. Seated (L–R): W. Nernst, M. Brillouin, E. Solvay, H. Lorentz, E. Warburg, J. Perrin, W. Wien, M. Curie, and H. Poincaré. Standing (L–R): R. Goldschmidt, M. Planck, H. Rubens, A. Sommerfeld, F. Lindemann, M. de Broglie, M. Knudsen, F. Hasenöhrl, G. Hostelet, E. Herzen, J. H. Jeans, E. Rutherford, H. Kamerlingh Onnes, A. Einstein, and P. Langevin. Almost half of the participants had won or would win a Nobel prize.

# §2 Phase space

Before we can understand the general definition of an 'adiabatic invariant', there are a few concepts that I need to introduce. First, it will prove very useful to visualise the motion of the system in an abstract space called *phase space*.

For simplicity, I will focus on systems with only one degree of freedom. The state of such a system, at a given time $t$, is described by a generalised coordinate, $q$, and the associated canonical momentum, $p$. In mechanics problems, $p$ is typically a function of $q$, its rate of change $\dot{q}$, and possibly $t$. If we know the Lagrangian $\mathcal{L}(q, \dot{q}, t)$ for our system, then $p$ is given by $p = \partial\mathcal{L}/\partial\dot{q}$. We can visualise the state of the system by plotting the point $(q, p)$; the two-dimensional space that this point lives in is called 'phase space'.

As the system evolves, the values of $q$ and $p$ describing the state of the system change. Thus, the point $(q, p)$ moves around, tracing out a curve in phase space. For an

[‡]This problem can also be solved directly, without adiabatic invariants; for example, see Gnädig *et al.* (2016), problem 23; or, check out the original treatment in Rayleigh (1902).

oscillating system whose parameters *don't* change with time, the point $(q, p)$ moves repeatedly around a closed curve. This curve has equation $\mathcal{H}(q, p) = \text{const.}$, where $\mathcal{H}(q, p)$ is the Hamiltonian; for mechanical systems, this is usually just the equation of energy conservation.

Let's see what this looks like for a pendulum. A natural choice of generalised coordinate is the angle $\theta$ that the string makes with the vertical. Then, the Lagrangian is $\mathcal{L}(\theta, \dot{\theta}, t) = \frac{1}{2}m\dot{\ell}(t)^2 + \frac{1}{2}m\ell(t)^2\dot{\theta}^2 + mg\ell(t)\cos\theta$, so the corresponding canonical momentum is $L = \partial\mathcal{L}/\partial\dot{\theta} = m\ell(t)^2\,\dot{\theta}$. This is just the angular momentum of the bob about the pivot point where the string meets the ceiling. If the length $\ell$ is fixed, then the Hamiltonian is $\mathcal{H}(\theta, L) = L^2/2m\ell^2 - mg\ell\cos\theta$, so the system moves around the constant-energy curve $L^2/2m\ell^2 - mg\ell\cos\theta = \text{const.}$ These curves in phase space are plotted in figure 4.

Each constant-energy curve corresponds to a possible motion of the pendulum. The amplitude of the motion can be read off from the size of the variation of the $\theta$ coordinate along the horizontal axis. We can see that these curves come in three different types. First, for amplitudes less than $\pi$, the curves are closed and roughly elliptical; these curves represent the pendulum swinging back and forth. Second, there is one curve for which the amplitude is exactly $\pi$; this curve is called the *separatrix* and it represents the pendulum *just* managing to swing until it points vertically upwards. Third, at even higher energies, the curves are no longer closed; these curves represent the pendulum swinging around and around the pivot in one direction without stopping, either clockwise or anti-clockwise according to the sign of $L$. This kind of motion is called 'circulation'.

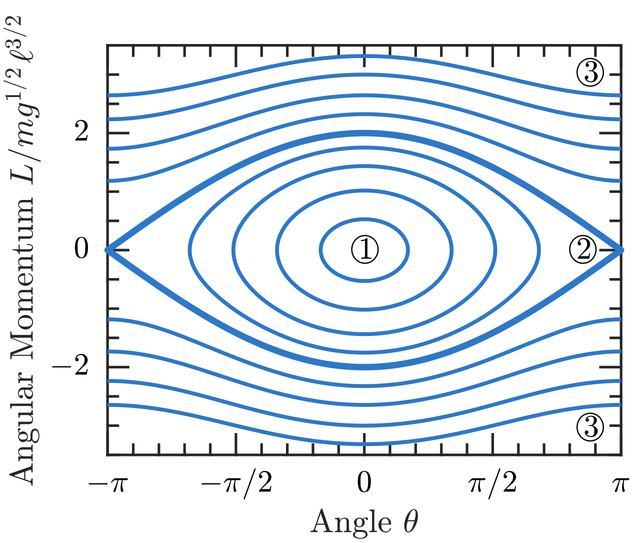


Figure 4: Constant-energy curves in phase space for a pendulum, plotted against angle and angular momentum (normalised to be dimensionless). In region ①, the pendulum motion is a back-and-forth oscillation. The thicker curve labelled ② is the separatrix. In each region labelled ③, the pendulum motion is circulation in a particular direction around the pivot.

We could add a third dimension to our phase-space plots by including a time axis. The resulting three-dimensional space is called *extended phase space*. In extended phase space, the state of the system at a given moment is represented by a point $(q, p, t)$. This point moves around as the system evolves; it traces out a spiral, as shown in figure 5. It will be useful to have a way of referring to the curve traced out by the point $(q, p, t)$ as it moves; we'll call it the *phase trajectory*, or just the trajectory, of the system.

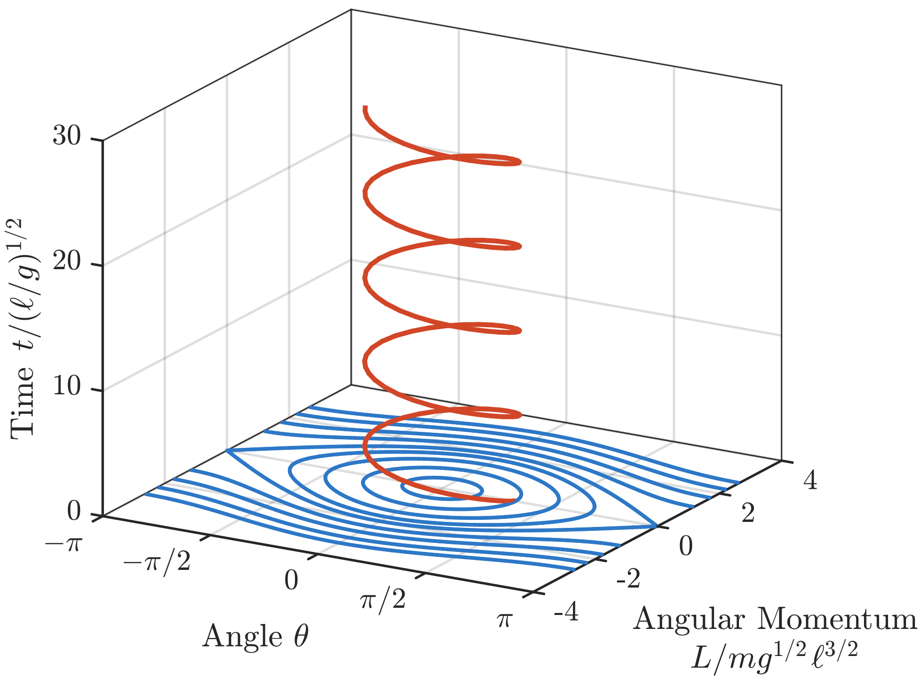


FIGURE 5: Trajectory (red) of a pendulum through extended phase space, for a particular set of initial conditions. In the $t = 0$ plane, the phase portrait from figure 4 is drawn (blue). Since the trajectory passes through a constant-energy curve inside the separatrix, the energy of the pendulum is low enough that its motion is a back-and-forth oscillation with amplitude less than $\pi$.

## §3 Adiabatic invariants

Now let's think about what happens when we start slowly varying a parameter or, more specifically, when we slowly vary the Hamiltonian. In our three-dimensional extended-phase-space plot, suppose we normalise the time coordinate along the vertical axis to $T_\ell$, the timescale for the slow variation. Then, since $T(t) \ll T_\ell$, the phase trajectory will spiral very rapidly around the vertical line $\theta = L = 0$, as in figure 6.

This phase trajectory is analogous to the field line in problem 3 of EuPhO 2025. As in that problem, the spiralling trajectory seems to trace out a tubular surface, which I'll call $S$. A more precise definition of $S$ is as follows. Let the initial energy of the pendulum be $E_0$. In the $t = 0$ plane, let $C$ be the curve $\mathcal{H}(q, p, t = 0) = E_0$.

Through every point of $C$, we can draw a phase trajectory. The set of all possible phase trajectories through points of $C$ forms a surface; this surface is the one I define to be $S$. Each phase trajectory in this set traces out a tight spiral, as described above. Therefore, $S$ looks like a tube whose cross section, at any fixed $t$, approximately coincides with a constant-energy curve $\mathcal{H}(q, p, t) = \text{const.}$, like the curves plotted in figure 4. The cross section has this shape because the condition $T(t) \ll T_\ell$ means the system barely changes over one oscillation, so the phase trajectory cannot deviate much from the shape it would have if there were no time variation. Over many oscillations, however, the energy of the pendulum could change, causing the tube to grow or shrink (see figure 6).

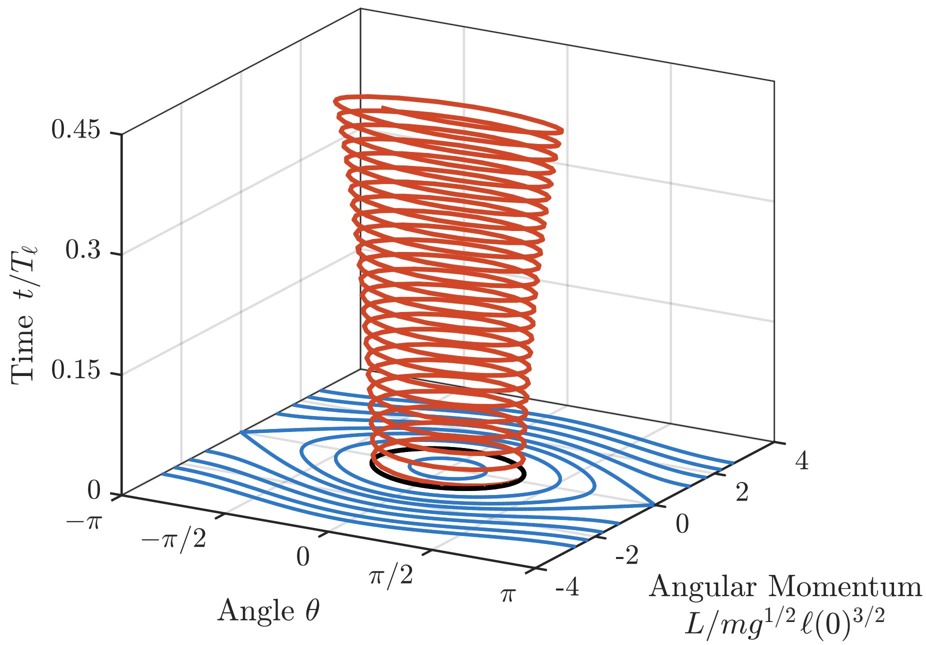


Figure 6: Trajectory (red) through extended phase space of a pendulum with a slowly varying length. For this simulation, I set $\mathrm{d}\ell/\mathrm{d}t = -0.003\,g^{1/2}\ell(0)^{1/2}$. The time coordinate is normalised to $T_\ell$, which means the trajectory forms a tight spiral. In the $t = 0$ plane, the phase portrait from figure 4 is drawn (blue). The curve $C$ is highlighted (black).

The key idea in the solution to problem 3 was that the magnetic flux through the middle of surface $S$ was conserved. Is there an analogous conservation law for general Hamiltonian systems? It turns out there is! Let's start with the equations defining the phase trajectory: these are Hamilton's equations,

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial p}, \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial q}. \tag{1}$$

We can rephrase these equations as a single vector equation for the motion of the point $(q, p, t)$ through extended phase space:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q \\ p \\ t \end{pmatrix} = \begin{pmatrix} \partial\mathcal{H}/\partial p \\ -\partial\mathcal{H}/\partial q \\ 1 \end{pmatrix} . \tag{2}$$

This is extremely similar to the equation for a magnetic field line $(x(s), y(s), z(s))$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} , \tag{3}$$

where $s$ is a parameter describing position along the field line. I'll define $\boldsymbol{X}$ to be the vector field on the right side of (2). Then, $\boldsymbol{X}$ is analogous to the magnetic field $\boldsymbol{B}$: it tells us the direction of the phase trajectory though any point in extended phase space, similar to how $\boldsymbol{B}$ tells us the direction of the magnetic field line through any point in physical space.

Recall that magnetic flux conservation is a consequence of Gauss's Law, which states that the divergence of $\boldsymbol{B}$ is zero:

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 . \tag{4}$$

Crucially, $\boldsymbol{X}$ satisfies its own version of Gauss's Law. The identity

$$\frac{\partial}{\partial q}\left(\frac{\partial\mathcal{H}}{\partial p}\right) + \frac{\partial}{\partial p}\left(-\frac{\partial\mathcal{H}}{\partial q}\right) + \frac{\partial}{\partial t}(1) = 0 , \tag{5}$$

which is satisfied thanks to the commutativity of partial derivatives, is exactly analogous to (4) — I've just replaced the components of $\boldsymbol{B}$ with components of $\boldsymbol{X}$, and I've replaced $x$, $y$ and $z$ with the coordinates $q$, $p$ and $t$ that describe position in extended phase space. Thus, (5) tells us that the divergence of $\boldsymbol{X}$ in extended phase space is zero. This result is called *Liouville's Theorem*. More precisely, Liouville's Theorem states that the phase trajectories for a Hamiltonian system are the streamlines for an incompressible flow.[¶]

In the same way that Gauss's Law (4) implies that the magnetic flux through $S$ in problem 3 is conserved, Liouville's Theorem (5) implies that the flux of $\boldsymbol{X}$ through $S$ is conserved for general Hamiltonian systems. There is no flux out through the walls of $S$ because —– by construction — $\boldsymbol{X}$ is tangent to $S$ at every point.

Let's use this conservation law by computing the flux of $\boldsymbol{X}$, contained within tube $S$, through a given plane of constant $t$. I have already argued that the intersection of $S$ and plane of constant $t$ must be, approximately, a curve of the form $\mathcal{H}(q, p, t) = \text{const}$. Since the component of $\boldsymbol{X}$ normal to the constant-$t$ plane is simply 1, the flux of $\boldsymbol{X}$ contained within tube $S$ is just the phase-space area of the cross section,

$$J(E, t) = \int p\,\mathrm{d}q . \tag{6}$$

[¶]Liouville's Theorem also holds for systems with more than one degree of freedom.

Here, the integration limits are chosen so that we are working out the phase-space area enclosed by the constant-energy curve with energy $E$, *viz.*, $\mathcal{H}(q, p, t) = E$, as shown in figure 7.

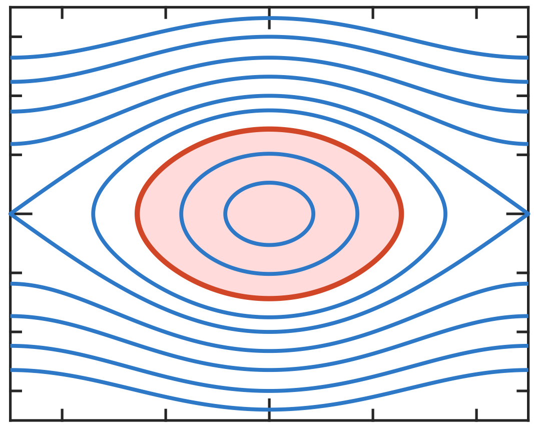


Figure 7: Phase-space area (shaded in red) enclosed by a constant-energy curve $\mathcal{H}(q, p, t) = E$. This area is the adiabatic invariant $J(E, t)$.

The quantity $J$ in (6) is called the adiabatic invariant. It is a function of the energy $E$ of the system and the time $t$ because the integration curve depends on $E$ and $t$. Conservation laws are some of the most powerful tools we have for solving physics problems; constancy of the adiabatic invariant provides us with a new conservation law for slowly varying systems. In the one-dimensional systems discussed here, conservation of $J$ makes it easy to determine how the energy evolves with time.

Note that the adiabatic invariant is an approximation to an exactly conserved quantity (namely, the flux of $\boldsymbol{X}$ through $S$). The reason we use $J$ instead of the exact flux is that the exact shape of $S$ is complicated and can only be found by solving the time-dependent equations of motion of the system, which is precisely what we want to avoid doing![‖]

**! Warning**

The adiabatic invariant is often described as 'approximately conserved'. Because of this, it's tempting to think that the time derivative of the adiabatic invariant must be smaller than the time derivative of a quantity that isn't 'approximately conserved', like the energy. In fact, this isn't true! The time derivative of the energy, at any instant, has characteristic size $\mathrm{d}E/\mathrm{d}t \sim E/T_\ell$. It turns out that the time derivative of the adiabatic invariant is also $\mathrm{d}J/\mathrm{d}t \sim J/T_\ell$. (We can prove this by differentiating the definition of $J$ and using Hamilton's equations

[‖] A more advanced idea, which I won't discuss any further in these notes, is that it is possible to find higher-order corrections to (6) in order to approximate the exactly conserved flux more accurately. For more details and history, see the review by Henrard (1993).

of motion.) In what sense, then, is $J$ 'approximately conserved' while $E$ is not? The answer can be understood from the time traces of $E$ and $J$ plotted in figure 8. This plot shows that, whereas $E$ slowly drifts upwards over long times, $J$ just oscillates near its initial value. The energy changes relative to its initial value by $\Delta E \sim E$, which is not small. Meanwhile, during the variation, the adiabatic invariant $J$ only changes by $\Delta J \sim (T/T_\ell)\, J$, which is small; this is actually one way of *defining* adiabatic invariants for more general systems. In other words, *changes in $J$ do not accumulate over long times*; this is what makes $J$ special.

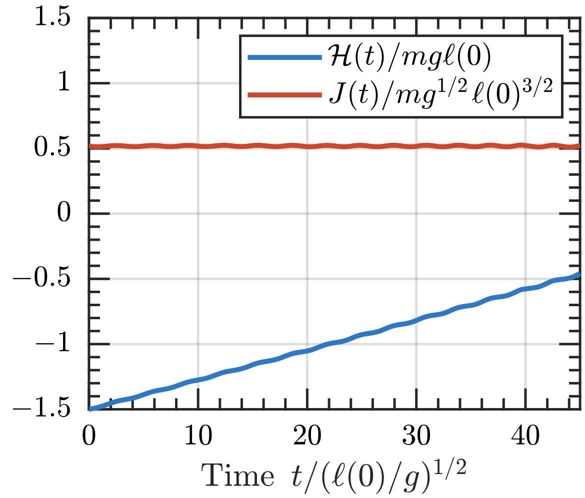


FIGURE 8: Plot of energy and the adiabatic invariant (both normalised to be dimensionless) over time for a pendulum with slowly varying length. For this simulation, I set $\mathrm{d}\ell/\mathrm{d}t = -0.01\,g^{1/2}\ell(0)^{1/2}$. Both $\mathcal{H}$ and $J$ 'wobble' on the timescale of an oscillation; these wobbles are the reason $\mathrm{d}J/\mathrm{d}t \sim J/T_\ell$ is nonzero. Over long times, however, $J$ remains close to its initial value.

## §4 Examples

For practice, let's compute $J$ for two simple systems: a harmonic oscillator and a pendulum.

**Example**

The simplest possible example is a harmonic oscillator with a slowly varying frequency. The Hamiltonian for this system is $\mathcal{H}(x,p,t) = p^2/2m + m\omega(t)^2x^2/2$. Let the timescale for the variation be $T_\omega$; then, $T_\omega^{-1} \sim (1/\omega)(\mathrm{d}\omega/\mathrm{d}t)$. For the variation to be slow, we need $T_\omega \gg 1/\omega$, or $\mathrm{d}\omega/\mathrm{d}t \ll \omega^2$, for all $t$. When this assumption is satisfied, we can use conservation of the adiabatic invariant, which is the phase-space area inside a constant-energy curve. For the harmonic-oscillator Hamiltonian, these curves are elliptical, as shown in figure 9. Using the formula for the area of an ellipse, we find

$$J = \frac{2\pi E}{\omega}\,. \tag{7}$$

So, if the oscillator varies slowly, the ratio of its energy to its frequency is constant. This is what Einstein stated in 1911; of course, the answer is also valid for a pendulum when its amplitude is small.

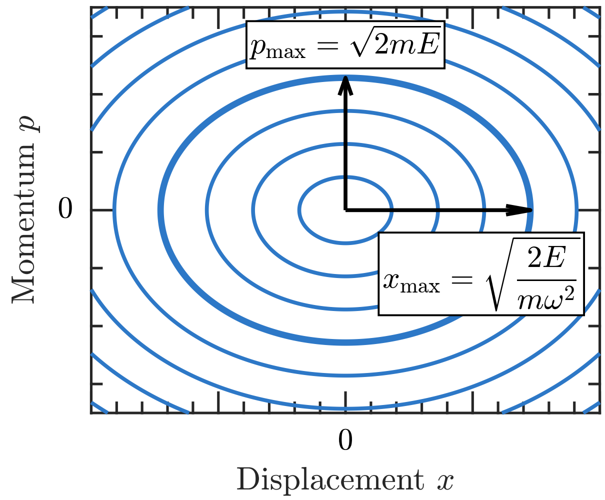


FIGURE 9: Elliptical constant-energy curves in phase space for a simple harmonic oscillator. The lengths of the semi-major and semi-minor axes are shown.

**Example**

For a more complicated example, let's return to pendulum problem — this time I won't assume the amplitude is small. The adiabatic invariant is $J = \oint L\,\mathrm{d}\theta$, where the integral gives the area enclosed by the curve $L^2/2m\ell^2 - mg\ell\cos\theta = E$. Let's evaluate this integral for an oscillating pendulum whose amplitude is less than $\pi$ (the integrals are a bit different for a circulating pendulum). Define $E = -mg\ell(1-2\mathcal{E})$, so $\mathcal{E}$ is a dimensionless measure of the energy: when $\mathcal{E}\to 0$, the pendulum amplitude becomes small, and when $\mathcal{E}\to 1$, the pendulum approaches the separatrix. Writing out the definition of $J$, we have

$$J = 4\sqrt{2}mg^{1/2}\ell^{3/2}\int_0^{\theta_{\max}}\sqrt{\cos\theta - (1-2\mathcal{E})}\,\mathrm{d}\theta\,, \tag{8}$$

where $\theta_{\max}$ satisfies $\cos\theta_{\max} = 1-2\mathcal{E}$. You could give this integral to Mathematica or ChatGPT or whatever, but — in case you want to be able to solve such integrals while stranded on a desert island — I'll explain one way of doing it by hand.

The integral would look nicer if both integration limits were constants, independent of $\mathcal{E}$. I'll arrange for this to be the case in two steps. First, I'll use a substitution that replaces the trigonometric functions with simpler functions. Then, I'll use a linear shift to fix the integration limits to be 0 and 1.

For the first step, I substitute $u = \sin^2(\theta/2)$. This substitution is very common because of the identities

$$\sin\theta = 2\sin(\theta/2)\cos(\theta/2) = 2\sqrt{u(1-u)}\,, \tag{9a}$$

$$\cos\theta = 1 - 2\sin^2(\theta/2) = 1 - 2u\,, \tag{9b}$$

$$\mathrm{d}\theta = \frac{\mathrm{d}u}{\sqrt{u(1-u)}}\,, \tag{9c}$$

which show that the trigonometric functions turn into simple combinations of $u$ and $1-u$. The result is

$$J = 8mg^{1/2}\ell^{3/2}\int_0^{\mathcal{E}} \sqrt{\frac{\mathcal{E}-u}{u(1-u)}}\,\mathrm{d}u\,. \tag{10}$$

Next, I use $w = u/\mathcal{E}$ to fix the integration limits:

$$J = 8mg^{1/2}\ell^{3/2}\mathcal{E}\int_0^1 \sqrt{\frac{1-w}{w(1-\mathcal{E}w)}}\,\mathrm{d}w\,. \tag{11}$$

Finally, motivated by (13), I'll return to a trigonometric form by substituting $w = \sin^2\varphi$, which gives

$$\begin{aligned} J &= 16mg^{1/2}\ell^{3/2}\mathcal{E}\int_0^{\pi/2} \frac{\cos^2\varphi}{\sqrt{1-\mathcal{E}\sin^2\varphi}}\,\mathrm{d}\varphi \\ &= 16mg^{1/2}\ell^{3/2}\int_0^{\pi/2}\left(\sqrt{1-\mathcal{E}\sin^2\varphi} - \frac{1-\mathcal{E}}{\sqrt{1-\mathcal{E}\sin^2\varphi}}\right)\mathrm{d}\varphi \\ &= 16mg^{1/2}\ell^{3/2}\left[E(\mathcal{E}) - (1-\mathcal{E})K(\mathcal{E})\right]\,, \end{aligned} \tag{12}$$

where

$$K(m) = \int_0^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1-m\sin^2\varphi}} \tag{13a}$$

$$E(m) = \int_0^{\pi/2} \sqrt{1-m\sin^2\varphi}\,\mathrm{d}\varphi \tag{13b}$$

are special functions called the complete elliptic integrals of the first and second kind, respectively. Formula (12) is what I used to create the plot of $J$ in figure 8. It is not hard to check that, as $\mathcal{E}\to 0$, we recover the result for a simple harmonic oscillator; we need the identities $K(m)\to(\pi/2)(1+m/4)$ and $E(m)\to(\pi/2)(1-m/4)$ as $m\to 0$, which can be derived by Taylor expanding the integrands in the definitions (13) for small $m$.

**Warning**

For the adiabatic invariant to actually be invariant, we needed Liouville's Theorem, which only applies if the mechanical system is described by a Hamiltonian. This means, for example, we cannot assume adiabatic-invariant conservation will still apply when frictional damping forces are present.

**Final warning**

Sometimes, slowly changing a parameter causes the phase trajectory to cross the separatrix. The period for the separatrix orbit is infinite, because it takes an infinite time for the pendulum to swing all the way up to the vertical (unstable) equilibrium point. This means that, around the separatrix, the adiabatic assumption $T_\ell \gg T(t)$ must fail! The change in the adiabatic invariant due to this separatrix crossing (compared with what we would have if it were truly conserved) can be calculated and is usually small. This difficult calculation was carried out for the pendulum by Timofeev (1978), and for general Hamiltonian systems by Cary *et al.* (1986) and Neishtadt (1987).

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