ROU-S3 T3

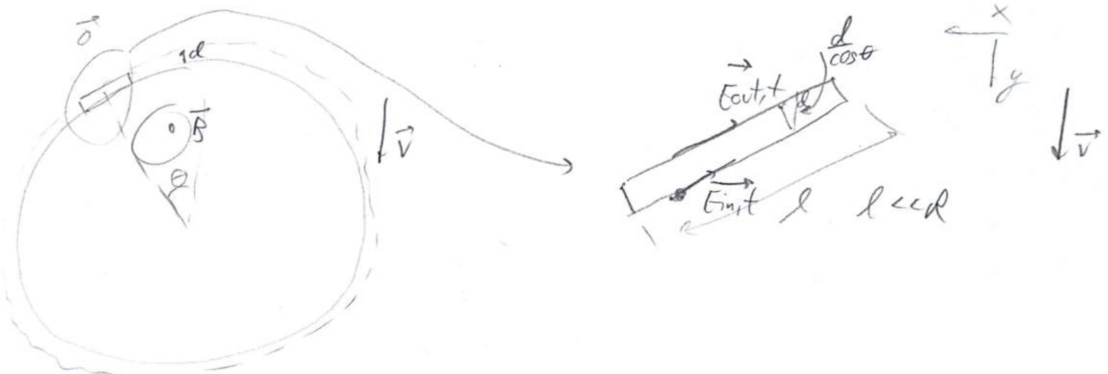
Cover Sheet for Solutions

7. EuPhO 23
Hannover Germany

Leibniz Universität Hannover

IPN
Leibniz Institute for Science and Mathematics Education

• The change of the magnetic field will lead to an induced curl of $\vec{E}$ in the plate at the edge of the region with a field. Let us consider that the "edge" of the region is actually a very thin ring, of width $d \ll R$. (see Fig.) I will consider that the spatial magnetic field rises, ~~at a~~ or decreases, at a constant ~~sp~~ rate within this region.



• We have

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad |\nabla \times \vec{E}| = \frac{dB}{dt} \cancel{= \frac{dB}{dy} \cdot \frac{dy}{dt}}$$

• The time needed for the ring element at angle θ to fully enter the field is

$$t = \frac{d/\cos\theta}{v} = \frac{d}{v\cos\theta},$$

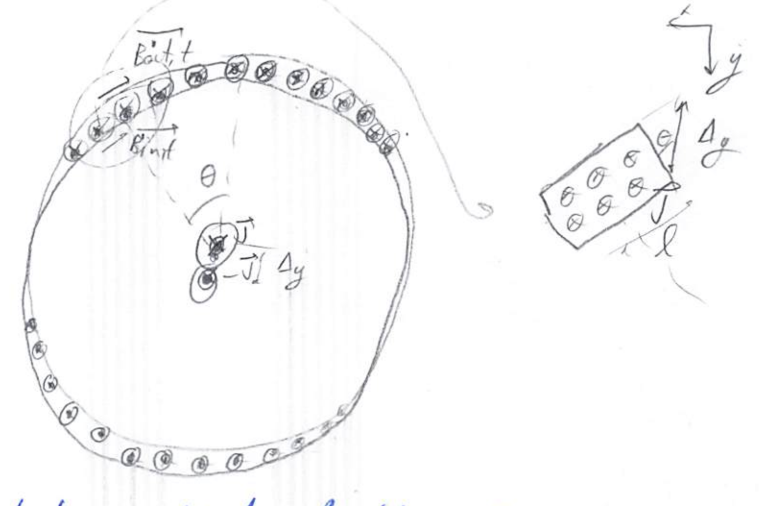
so $$\frac{dB}{dt} = \frac{B}{t} = \frac{Bv\cos\theta}{d},$$

and the definition of the curl implies that

$$\oint \vec{E} \cdot d\vec{l} = \iint \nabla \times \vec{E}\, ds \Rightarrow \cancel{E}\ \cancel{\Delta E_t} (E_{out,t} - E_{in,t}) \cdot l = dl \cdot |\nabla \times \vec{E}|$$

$$\Leftrightarrow \boxed{\Delta E_t = Bv\cos\theta} \qquad (1)$$

• We can solve this problem (of finding $\vec{E}$ in the whole space) by analyzing another simpler, mathematically analogous problem. This is the following situation: two infinite cylinders, each of radius R, carry the same homogenous current density $\vec{J}$ in opposite directions. They are superposed, with their centres a very small (infinitesimal) distance Δ_y apart along the y-axis. I will prove that the magnetic field B' of this situation is analogous to the electric field in the current problem.



• Ampere's law states that, at the edge,

$$\Delta B'_t = \mu_0 \mathcal{J} \quad (\text{which is derived from } \Delta B'_t \cdot l = \mu_0 I = \mu_0 \mathcal{J} l)$$

where $\mathcal{J}$ is the surface current density, ~~equal to~~ (see fig.)

$$\cancel{\mathcal{J} =} \quad \mathcal{J} = \frac{1}{l} \cdot \frac{J l \cdot (\Delta_y \cos\theta)}{l} = J \Delta_y \cos\theta, \text{ so}$$

$$\boxed{\Delta B'_t = \mu_0 J \Delta_y \cos\theta} \qquad (2)$$

Note: I considered the width of the edge region where the two wires do not overlap and hence there is net current. In the rest of the wires, there is no net current.

• Outside the edge space, this field ~~is~~ has divergence and curl 0, and is bidimensional (since it is constant along the z-axis). The divergence is also 0 at the edge, since $\nabla \cdot \vec{B}' = 0$.

• In the case of the metal sheet, the component of $\vec{E}$ within the sheet has divergence 0 everywhere, due to the continuity equation $\nabla \cdot \vec{J} = 0$, and $\vec{J} = \frac{\vec{E}}{\rho}$. Also, its curl outside the edge is 0, since ~~$\nabla \times \vec{E} = \vec{0}$~~

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},$$

and $\frac{\partial \vec{B}}{\partial t}$ is nonzero only at the edges. This field is also two-dimensional. This proves that the two fields are equivalent, up to the factors that appear in Eqs. (1) and (2).

• Hence, if we find $\vec{B}'(x,y)$, we can transform this to $\vec{E}$ using

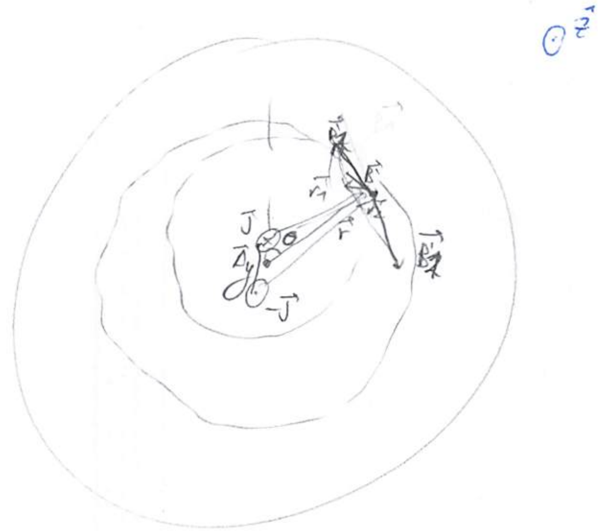
$$\boxed{\vec{E} = \vec{B}' \cdot \frac{Bv}{\mu_0 J \Delta y}}$$

a) • The field of ~~an current long~~ infinite wire that carries current I, at a distance r from the axis of the wire, is

$$\vec{B}' = \frac{\mu_0 I}{2\pi r} \hat{\theta},$$

where $\hat{\theta}$ is a tangential unit vector.

- Inside the two wires, the resultant field is given by (see fig.)



$$\vec{B'} = \vec{B'_1} + \vec{B'_2},$$

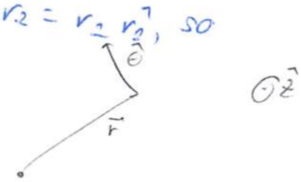
with

$$\vec{B'_1} = -\frac{\mu_0 I_1}{2\pi r_1}\hat{\theta_1} = -\frac{\mu_0 \cdot \pi r_1^2 J}{2\pi r_1}\hat{\theta_1} = -\frac{\mu_0 r_1 J}{2}\hat{\theta_1};$$

$$\vec{B'_2} = +\frac{\mu_0 I_2}{2\pi r_2}\hat{\theta_2} = +\frac{\mu_0 \pi r_2^2 J}{2\pi r_2}\hat{\theta_2} = +\frac{\mu_0 r_2 J}{2}\hat{\theta_2},$$

where I used the fact that only the ~~field~~ current inside the radius r_i $(i=1,2)$ for each wire causes ~~elect~~ magnetic field at radius r_i from the axis.

- We have $\hat{\theta_1} = \hat{z}\times\hat{r_1}$, $\hat{\theta_2} = \hat{z}\times\hat{r_2}$, and $\vec{r_1} = r_1\hat{r_1}$, $\vec{r_2} = r_2\hat{r_2}$, so



$$\vec{B'} = \frac{\mu_0 J}{2}(-r_1\hat{\theta_1} + r_2\hat{\theta_2})$$
$$= \frac{\mu_0 J}{2}(-r_1\hat{z}\times\hat{r_1} + r_2\hat{z}\times\hat{r_2})$$
$$= \frac{\mu_0 J}{2}\hat{z}\times(-\vec{r_1} + \vec{r_2})$$
$$= \frac{\mu_0 J}{2}\hat{z}\times(-\Delta_y\hat{y})$$
$$= \frac{\mu_0 J}{2}\Delta_y\hat{x} \Rightarrow \vec{B'} \text{ is constant inside the wires and}$$

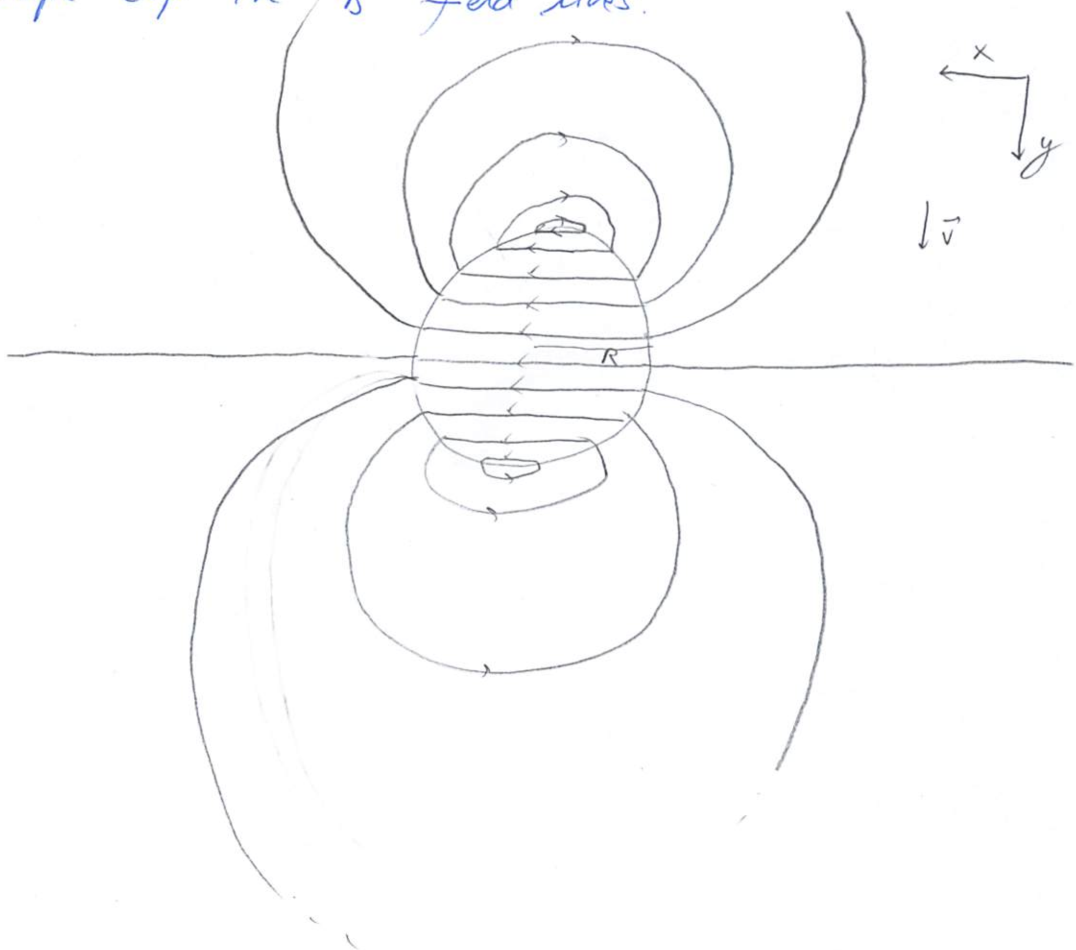
has the value $\frac{\mu_0 J \Delta_y}{2}\hat{x}$.

• Outside the wires, the field $\vec{B'}$ will have a shape somewhat similar to that of a dipole field. Since $\vec{J} = \frac{1}{\rho}\vec{E}$, and $\vec{E} = \frac{Bv}{\mu_0 j \Delta y} \cdot \vec{B'}$, we find that

$$\vec{J} = \frac{Bv}{\mu_0 j \Delta y \rho} \vec{B'},$$

(here J is considered in the magnetic infinite wire model, and $\vec{J}$, in the actual problem)

i.e. $\vec{J}$ and $\vec{B'}$ are also equivalent, so the shape of the current stream lines is the same as the shape of the $\vec{B'}$ field lines.

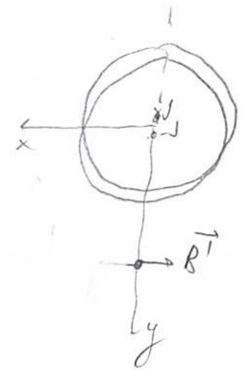


b) • ~~For~~ Let ~~y be~~ the origin of the coordinate system be the intersection of the metal plate and the axis of the magnets. We know that, for $-R \le y \le R$, $\vec{B'} = \frac{\mu_0 J \Delta y}{2}\hat{x}$. Hence,

$$\vec{J} = \frac{Bv}{2\rho}\hat{x}, \quad -R \le y \le R.$$

• For other values of y, we have:

$$\vec{B'} = \vec{B_1'} + \vec{B_2'}$$
$$= -\hat{x}\left(\frac{\mu_0 J \cdot \pi R^2}{2\pi(y-\frac{\Delta y}{2})} - \frac{\mu_0 J \pi R^2}{2\pi(y+\frac{\Delta y}{2})}\right); \ \Delta y \ll y;$$
$$\vec{B'} \approx -\hat{x} \cdot \frac{\mu_0 J R^2}{2y}\left(1 + \frac{\Delta y}{2y} - \left(1 - \frac{\Delta y}{2y}\right)\right)$$
$$= \cancel{-\hat{x}\frac{\mu_0 J \Delta y}{\pi y^2}} - \hat{x} \cdot \frac{\mu_0 J \Delta y R^2}{2y^2}$$



• So,

$$\vec{J} = -\frac{BvR^2}{2\rho y^2}\hat{x}, \quad |y| \ge R.$$

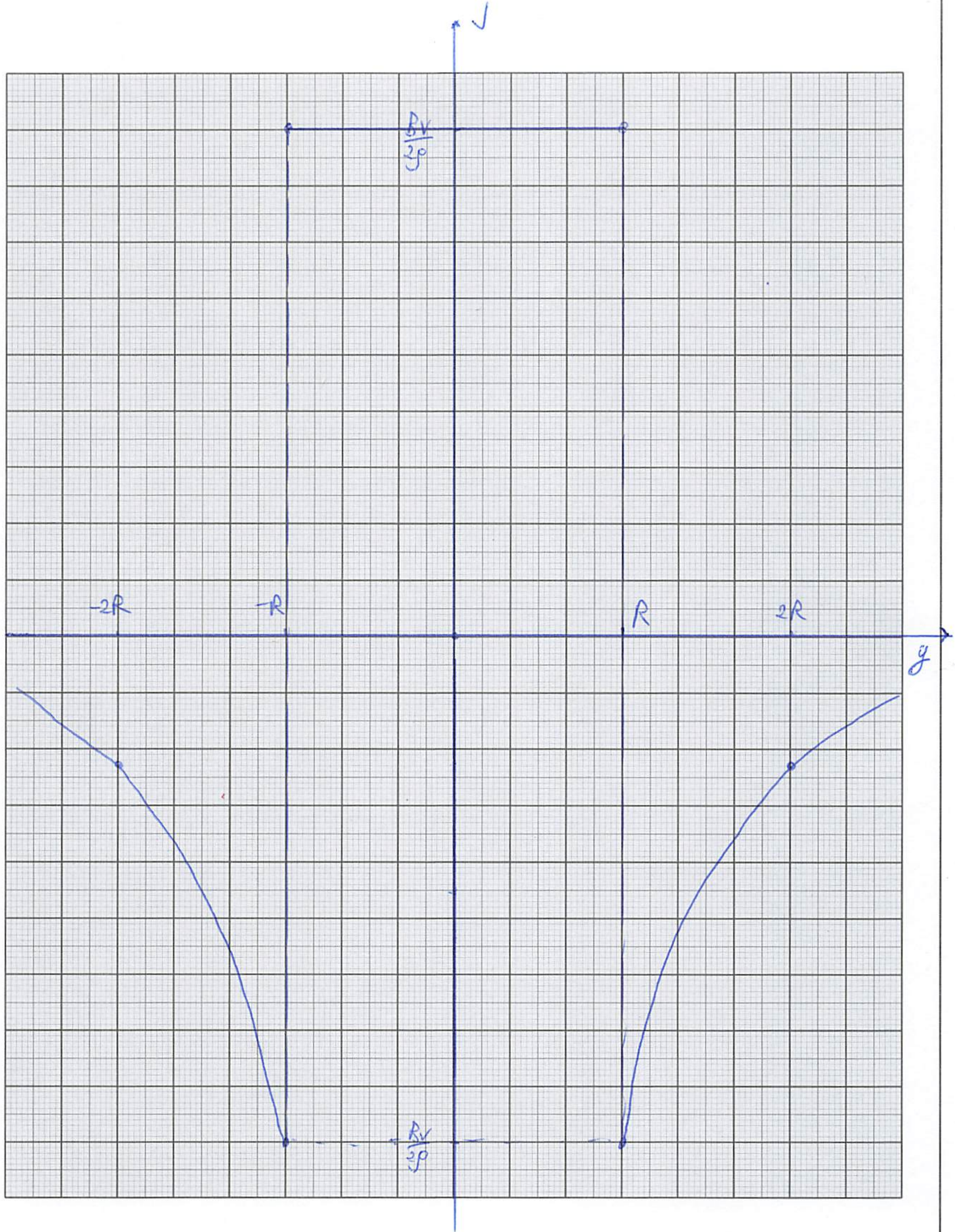
• Hence,

$$\boxed{\vec{J}(y) = \begin{cases} \frac{Bv}{2\rho}\hat{x}, & |y| \le R; \\ -\frac{Bv}{2\rho}\frac{y^2}{R^2}\hat{x}, & |y| \ge R; \end{cases}}$$

or, as an ~~magnitude~~ x-component,
$$J(y) = \begin{cases} \frac{Bv}{2\rho}, & |y| \le R; \\ -\frac{Bv}{2\rho}\frac{y^2}{R^2}, & |y| \ge R. \end{cases}$$

$$\cancel{J(y) = \begin{cases} \frac{Bv}{2\rho}, & |y| \le R; \\ \frac{Bv}{2\rho}\frac{y^2}{R^2}, & |y| \ge R \end{cases}}$$

The plot is on page 7.


J
$\frac{Bv}{2\rho}$
-2R
-R
R
2R
y
$-\frac{Bv}{2\rho}$

c) • The ~~forc~~ magnetic force on a current-carrying medium is $\vec{J} \times \vec{B}$ per unit volume. The only region of the plane in which such a force arises is that directly ~~beneath the~~ between the magnets. In this region, ~~the~~ $\vec{J}$ is constant, $\vec{J} = \frac{Bv}{2\rho}\hat{x}$, so the total force is

$$\vec{F} = \vec{J} \times \vec{B} \cdot V$$
$$= \frac{Bv}{2\rho}\hat{x} \times (\hat{z}B) \cdot (\pi R^2 \delta)$$
$$= -\frac{\pi B^2 v R^2 \delta}{2\rho}\hat{y}.$$

• So, the force required is $F = \frac{\pi B^2 v R^2 \delta}{2\rho}$, and it is oriented along the negative y-axis.