

• Since the observer is far away, the rays that will reach him ~~are~~ will be approximately parallel. In the figure above, the observer is to the left.

• In the figure above, point P is a short segment of the thread, which is very close to the line of vision passing through O (stick rod.) As can be seen, point P leads to two different emerging rays that are parallel to the line of sight OA; these two correspond to the two points of the loop, one closer to OA, the other one farther.

In what follows, I will prove that, generally, this may happen.

• According to Snell's law,

$$n \sin r = \sin i \quad (1)$$

Also,

$$2r = i + \theta$$

(outer angles), so  $r = \frac{i + \theta}{2}$ . (2)

• Using (2), (1) becomes

$$n \sin \frac{i+\theta}{2} = \sin i$$

$$\Leftrightarrow n \left( \sin \frac{i}{2} \cos \frac{\theta}{2} + \cos \frac{i}{2} \sin \frac{\theta}{2} \right) = 2 \sin \frac{i}{2} \cos \frac{i}{2} \quad (3)$$

• Now, we have assumed that  $1 < n < 1$  rad. This must be true, because otherwise the image of the very thin thread wouldn't be so strongly deformed. In this case, we can use the approximations  $\sin \frac{\theta}{2} \approx \frac{\theta}{2}$ ,  $\cos \frac{\theta}{2} \approx 1$  to rewrite (3) as

$$n \left( \sin \frac{i}{2} + \frac{\theta}{2} \cos \frac{i}{2} \right) = 2 \sin \frac{i}{2} \cos \frac{i}{2}$$

$$\Leftrightarrow \theta = \frac{4}{n} \sin \frac{i}{2} - 2 \operatorname{tg} \frac{i}{2} \quad (4)$$

• Let us find whether ~~this function~~  $\theta$  has any extrema for  $i \in [0, \frac{\pi}{2}]$ . If this were to happen, then the derivative of  $\theta$  w.r.t.  $\frac{i}{2}$  must be 0 for some  $i \in [0, \frac{\pi}{2}]$ :

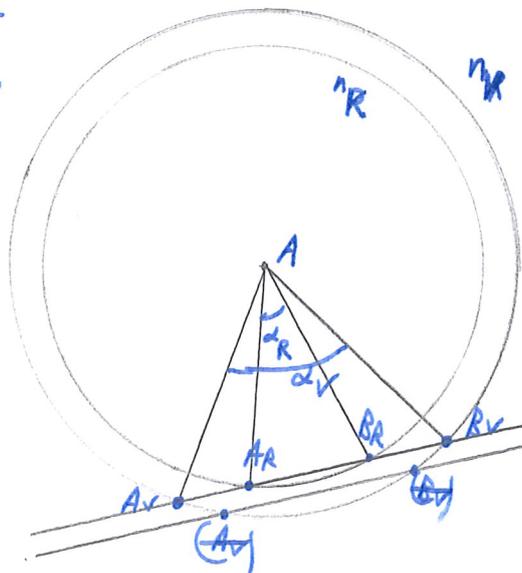
$$\frac{d\theta}{d(\frac{i}{2})} = \frac{4}{n} \cos \frac{i}{2} - \frac{2}{\cos^2 \frac{i}{2}} = 0$$

$$\Leftrightarrow \cos^3 \frac{i}{2} = \frac{n}{2}$$

• If  $\frac{n}{2} \geq \left(\frac{\sqrt{2}}{2}\right)^3 = \frac{1}{2\sqrt{2}} \Leftrightarrow n \geq \frac{1}{\sqrt{2}}$  (which is clearly true, then there indeed exists some  $i$  for which the value of  $\theta$  reaches <sup>and  $n \leq 2$</sup>  an extremum. This means that the derivative switches sign at that value of  $i$  and, hence, for all  $\theta <$  the maximum value, there may be two different values of  $i$  for which (4) is satisfied. This proves why the images of the thread take the shape of a loop.

• Let us now study further what the image formation process looks like. Since only for  $\theta \leq \theta_{max}$  images can be seen, this means that the area in which objects stuck to the sphere can be seen through it is a small circle of angular diameter  $\theta_{max}$ . This circle - which will further be called the object circle - is shown below for both refractive indices; the thread is also shown.

• We can see that the thread lies at the edge of the object circles; this is why its image is so strongly distorted. Only the short segment that lies within the circles is actually seen.



b. • For numerical computations, the approximate ( $\theta < 1$  rad) treatment above is no longer viable; we have to return to the general equation, (2), and use it.

• Let us study when does  $\theta$  reach a maximal value. If  $\theta$  is maximum, then  $\frac{d\theta}{di} = 0$ . Equation (2) tells us that

$$n \sin \frac{i+\theta}{2} = \sin i.$$

• Differentiating it, we get

$$\frac{n}{2} \cos \frac{i+\theta}{2} (di + d\theta) = \cos i di.$$

• But  $\frac{d\theta}{di} = 0$ , so

$$\frac{n}{2} \cos \frac{i+\theta}{2} = \cos i.$$

(5)

• So,

$$\begin{cases} (3) & n \sin \frac{i+\theta}{2} = \sin i \\ (5) & \frac{n}{2} \cos \frac{i+\theta}{2} = \cos i \end{cases} \Rightarrow (\sin i)^2 + (2 \cos i)^2 = n^2 \left( \sin^2 \frac{i+\theta}{2} + \cos^2 \frac{i+\theta}{2} \right) = n^2$$

$$\Rightarrow 3 \cos^2 i = n^2 - 1.$$

• Hence, the value of  $i$  for which the maximal value of  $\theta_{\text{max}}$  is reached,  $i_{\text{m}}$ , is given by

$$\cos^2(i_{\text{m}}) = \frac{n^2 - 1}{3}. \quad (5)$$

• From the given image, we can measure  $d$  (see the figure on the first page) and  $R$ ;  $\sin i = \frac{d}{R}$ , so we can measure  $i$ . The endpoints of the major axis of the ellipse that is the red image of the thread are correspond to  $i_{\text{m}}$ , because they are the furthest visible points ( $A_R$  and  $B_R$  on the figure on the third page).

• Hence, we can measure  $i$  for the rays that originate in  $A_R$  or  $B_R$ ; this is  $i_{\text{m}}$ , and hence, from (5),

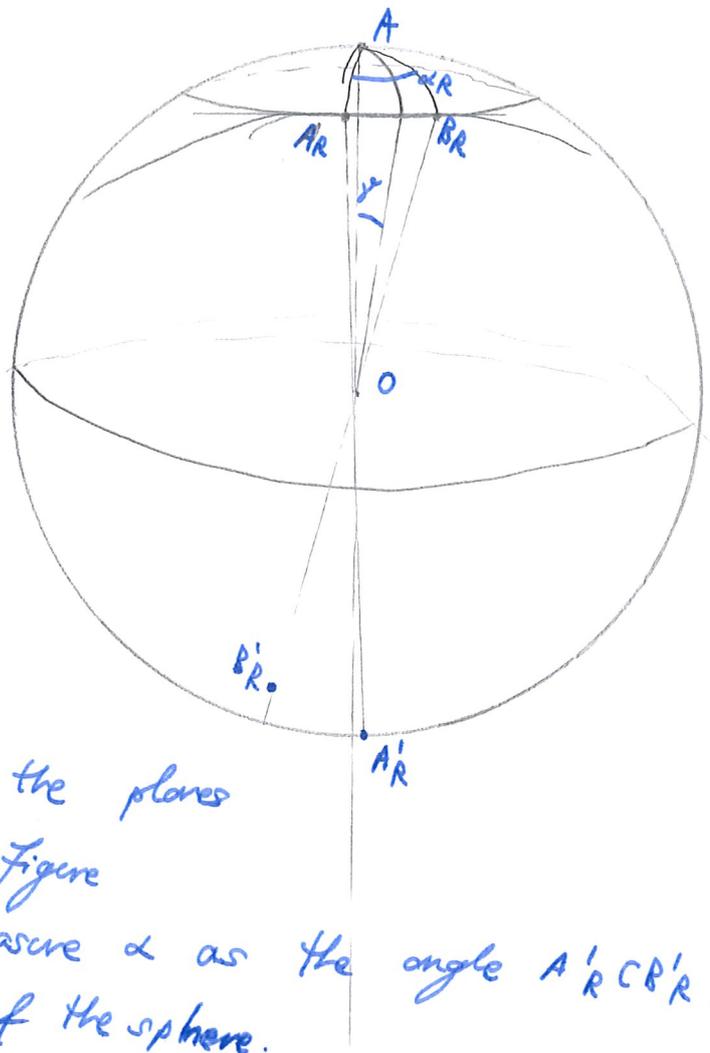
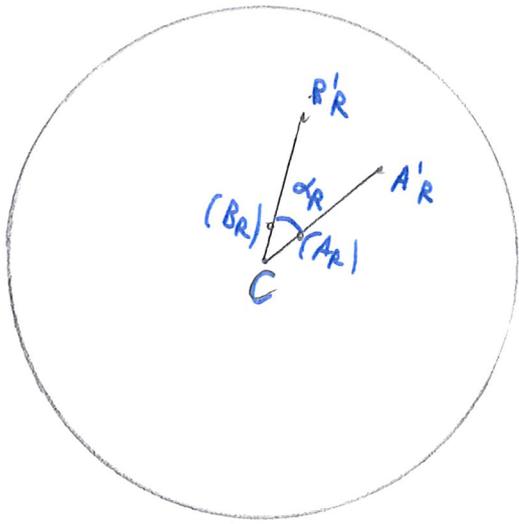
$$n_R = \sqrt{3 \cos^2(i_{\text{m}}) + 1}.$$

• Numerically,  $d = 7,1 \text{ cm}$ ,  $R = 9,5 \text{ cm}$ ,  $\sin i_{\text{m}} = i_{\text{m}} = \arcsin\left(\frac{d}{R}\right) = 48,4^\circ$ , and

$$\boxed{n_R = 1,52}$$

c. • We could in theory use the same method to find  $n_V$ , but in practice the difference is too small to be calculated this way.

• The image of  $A_R$  must be in the plane  $AOA_R$ . Let's call this image  $A'_R$ . Similarly,  $B'_R$  is in the plane  $BOB_R$ . This means that we can use the positions of  $A'_R$  and  $B'_R$  to find the positions of  $A_R$  and  $B_R$ . This will be explained in more detail in what follows.



• More clearly, the angle between the planes is  $\alpha_R$  (which was shown in the figure on page 3). Therefore, we can measure  $\alpha$  as the angle  $A'R C B'R$ , where C is the centre of the image of the sphere.

~~Let  $d$  be the (perpendicular) distance~~

• Let  $\delta$  be the angular distance between A and the closest part of the thread. We have  $\widehat{AAR} = \theta_R$  (the maximum  $\theta$  for which an image can be seen),  $\widehat{ABR} = \theta_R$ ; hence, the <sup>sine</sup> ~~cosine~~ law in the spherical triangles of interest tells us that

$$\cos \left( \frac{\widehat{ARBR}}{2} \right) = \cos \delta \cos \theta_R + \cos \frac{\alpha_R}{2} \sin \delta$$

$$\frac{\sin \left( \frac{\widehat{ARBR}}{2} \right)}{\sin \frac{\alpha_R}{2}} = \frac{\sin \delta}{\sin \theta_R}$$

$$\cos \left( \frac{\widehat{AAR}}{2} \right) = \cos \delta \cos \frac{\widehat{AAR}}{2}$$

$$\cos \theta_R = \cos \delta \cos \frac{\widehat{AAR}}{2} \cos \frac{\widehat{ARBR}}{2}$$

• But the sine law tells us that

$$\frac{\sin\left(\frac{\widehat{ARBR}}{2}\right)}{\sin\frac{\alpha_R}{2}} = \frac{\sin\widehat{ARR}}{1} = \sin\theta_R,$$

so  $\sin\left(\frac{\widehat{ARBR}}{2}\right) = \sin\theta_R \sin\frac{\alpha_R}{2}$ , from which

$$\cos\left(\frac{\widehat{ARBR}}{2}\right) = \sqrt{1 - \left(\sin\theta_R \sin\frac{\alpha_R}{2}\right)^2},$$

and

$$\begin{aligned} \cos\theta_R &= \cos^2\theta \cdot \sqrt{1 - \left(\sin\theta_R \sin\frac{\alpha_R}{2}\right)^2} \\ \Leftrightarrow (\cos^2\theta)^{-1} &= \frac{\left(1 - \left(\sin\theta_R \sin\frac{\alpha_R}{2}\right)^2\right)^{-1/2}}{\cos^2\theta_R} \end{aligned}$$

• In the same way, we get

$$(\cos^2\theta)^{-1} = \frac{1 - \left(\sin\theta_V \sin\frac{\alpha_V}{2}\right)^2}{\cos^2\theta_V}$$

• Equating the two expressions above,

$$\frac{1 - \left(\sin\theta_R \sin\frac{\alpha_R}{2}\right)^2}{\cos^2\theta_R} = \frac{1 - \left(\sin\theta_V \sin\frac{\alpha_V}{2}\right)^2}{\cos^2\theta_V}$$

•  ~~$\sin\theta_R$  and  $\theta$  are very close, so we can write  $\theta_V = \theta_R + \Delta\theta$ ,  $\alpha_V < \alpha_R$ , and write approximately that  $\sin\theta_V = \sin\theta_R + \Delta\theta \cos\theta_R$ ;  $\cos\theta_V \approx \cos\theta_R - \Delta\theta \sin\theta_R$ ; so that~~

~~$$\frac{1 - \sin^2\theta_R \sin^2\frac{\alpha_R}{2}}{1 - \sin^2\theta_V \sin^2\frac{\alpha_V}{2}} = \frac{\cos^2\theta_R}{\cos^2\theta_V}$$~~

~~$$\begin{aligned} \Leftrightarrow \frac{1 - \sin^2\theta_R \sin^2\frac{\alpha_R}{2}}{1 - (\sin^2\theta_R + 2\Delta\theta \sin\theta_R \cos\theta_R) \sin^2\frac{\alpha_V}{2}} &\approx \frac{\cos^2\theta_R}{\cos^2\theta_R - 2\Delta\theta \sin\theta_R} \end{aligned}$$~~

~~$$\Leftrightarrow -\frac{2\Delta\theta \sin\theta_R}{1} \approx -\frac{\sin^2\theta_R (\sin^2\frac{\alpha_V}{2} - \sin^2\frac{\alpha_R}{2})}{1 - \sin^2\theta_R \sin^2\frac{\alpha_R}{2}}$$~~

- We can express  $\theta_R$  or  $\theta_V$  by using the fact that

$$\begin{cases} (3) \quad n \sin \frac{i+\theta}{2} = \sin i \\ (5) \quad \frac{1}{2} \cos \frac{i+\theta}{2} = \cos i \end{cases} \Rightarrow \frac{n^2}{4} \left( \sin^2 \frac{i+\theta}{2} + \cos^2 \frac{i+\theta}{2} \right) = 1$$

$$(c) \quad 3 \sin^2 \frac{i+\theta}{2} = \frac{4}{n^2} = 1$$

So  $i_m + \theta = 2 \arcsin \left( \frac{1}{\sqrt{3}} \frac{2}{n} \right), \left( \sqrt{\frac{4}{n^2} - 1} / \sqrt{3} \right)$

$$\theta = 2 \arcsin \left( \frac{1}{\sqrt{3}} \frac{2}{n} \right) - i_m. \quad \theta = 2 \arcsin \left( \sqrt{\frac{4}{n^2} - 1} / \sqrt{3} \right) - i_m.$$

- For  $i_m = 49.4^\circ, n = n_R$ , we get

$$\theta_R = 50.5^\circ \quad \theta_R = 10.5^\circ.$$

- Measuring on the image, we find that

$$\begin{cases} \alpha_R = 23^\circ; \\ \alpha_V = 44^\circ. \end{cases}$$

~~so that, using the previous equation,~~

~~to~~

- For a simpler approximation, we can assume that the object circle lies in a plane. In these conditions, if  $d$  is the distance between the A and the threads, we have

$$AA_R^2 = d^2 + (AA_R \sin \frac{\alpha_R}{2})^2;$$

$$AA_V^2 = d^2 + (AA_V \sin \frac{\alpha_V}{2})^2; \text{ so}$$

$$AA_R \cos \frac{\alpha_R}{2} = AA_V \cos \frac{\alpha_V}{2}.$$

- But  $\frac{AA_R}{AA_V} = \frac{\theta_R}{\theta_V}$ , so that

$$\theta_V = \theta_R \frac{\cos(\alpha_R/2)}{\cos(\alpha_V/2)}$$

• Differentiating (3) and (5) w.r.t.  $n$ , we get

$$\begin{cases} \frac{dn}{n} \left( \sin \frac{i+\theta}{2} \right) + \frac{1}{2} (di+d\theta) \cos \frac{i+\theta}{2} n = \cos i di \\ \frac{dn}{n} \cos \frac{i+\theta}{2} - \frac{1}{4} n (di+d\theta) \sin \frac{i+\theta}{2} = -\sin i di \end{cases}$$

$$\Rightarrow \frac{dn}{n} \sin i + \frac{1}{2} (di+d\theta) n \cdot \frac{\cos i}{n} = \cos i di$$

$$\frac{1}{2} \frac{dn}{n} \cdot \cos i + \frac{1}{4} (di+d\theta) n \cdot \frac{\sin i}{n} = -\sin i di$$

$$\Rightarrow \begin{cases} \frac{dn}{n} \sin i + (di+d\theta) \cos i = \cos i di \Rightarrow d\theta \cos i + \frac{dn}{n} \sin i = 0 \\ \Rightarrow dn = -n \tan i d\theta \\ \frac{dn}{n} \cos i + \frac{1}{4} (di+d\theta) \sin i = -\sin i di \end{cases}$$

• Therefore,

$$\Delta n = -n \tan i \Delta \theta \text{ (approximately)}$$

• Numerically,  $\Delta \theta = \theta_1 - \theta_2 = \theta_2 \left( \frac{\cos \theta_2 (r)}{\cos (\alpha r/2)} - 1 \right) = 0,597^\circ = 1,04 \cdot 10^{-2} \text{ rad}$ , and

$$\Delta n = -1,78 \cdot 10^{-2}$$

